Harvey Mudd College Math Tutorial:

Computing Integrals by Substitution

Many integrals are most easily computed by means of a change of variables, commonly called a u-**substitution**.

Example

Let's compute $\int 2x(x^2-1)^4\,dx$ by making the substitution

$$\begin{aligned} u &= x^2 - 1 \\ du &= 2x\,dx. \end{aligned}$$

Then

$$\int 2x(x^2-1)^4\,dx = \int (x^2-1)^4(2x\,dx) = \int u^4\,du = \frac{u^5}{5} + C = \frac{(x^2-1)^5}{5} + C.$$

We may check this result by differentiating using the Chain Rule:

$$\frac{d}{dx}\left(\frac{(x^2-1)^5}{5} + C\right) = \frac{5(x^2-1)^4}{5}(2x) = 2x(x^2-1)^4. \qquad \checkmark$$

The substitution method amounts to applying the Chain Rule in reverse:

To compute $\int f(g(x))g'(x)\,dx$, we let

$$\begin{aligned} u &= g(x) \\ du &= g'(x)\,dx. \end{aligned}$$

Then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) = F(g(x))$$

where F is an antiderivative of f.

Example

To compute $\int \sin(2x)\cos(2x)\,dx$, let

$$\begin{aligned} u &= \sin(2x) \\ du &= 2\cos(2x)\,dx. \end{aligned}$$

Then

$$\int \sin(2x)\cos(2x)\,dx = \int \frac{1}{2}\sin(2x)[2\cos(2x)\,dx] = \int \frac{1}{2}u\,du = \frac{1}{4}u^2 + C = \frac{1}{4}\sin^2(2x) + C.$$

With practice, you will often be able to write down the result immediately.

Example

We can evaluate $\int \frac{dx}{(4x-3)^2}$ by letting

$$\begin{aligned} u &= 4x-3 \\ du &= 4\,dx \quad \longrightarrow \quad dx = \frac{1}{4}\,du. \end{aligned}$$

Then

$$\int \frac{dx}{(4x-3)^2} = \int \frac{\frac{1}{4}\,du}{u^2} = -\frac{1}{4u} + C = \frac{-1}{4(4x-3)} + C.$$

It is not always apparent until you try it whether or not a substitution will work.

Example

To compute $\int x\sqrt{x-3}\,dx$, we will try

$$\begin{aligned} u &= x-3 \quad \longrightarrow \quad x = u+3 \\ du &= dx. \end{aligned}$$

So

$$\begin{aligned} \int x\sqrt{x-3}\,dx &= \int (u+3)\sqrt{u}\,du = \int \left(u^{3/2} + 3u^{1/2}\right)\,du \\ &= \frac{2}{5}u^{5/2} + 2u^{3/2} + C = \frac{2}{5}(x-3)^{5/2} + 2(x-3)^{3/2} + C. \end{aligned}$$

We can also compute a definite integral using a substitution.

Example

Let's evaluate $\int_0^2 xe^{x^2}\,dx$. Let

$$\begin{aligned} u &= x^2 \\ du &= 2x\,dx. \end{aligned}$$

First, we will compute the *indefinite* integral:

$$\int xe^{x^2}\,dx = \int \left(\frac{1}{2}e^{x^2}\right)(2x\,dx) = \int \frac{1}{2}e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

Now we have two approaches for the *definite* integral:

Approach 1
Substitute back to
the original variable:
$\int xe^{x^2}\,dx = \frac{1}{2}e^u + C$
$= \frac{1}{2}e^{x^2} + C.$
So $\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2}\big|_0^2 \quad \Longrightarrow \quad \frac{1}{2}(e^4 - 1)$

Approach 2
Change the limits of integration:
Since $u = x^2$,
$u = 0$ when $x = 0$
and $u = 4$ when $x = 2$.
$\frac{1}{2}(e^4 - 1) \quad \Longleftarrow \quad \int_0^2 xe^{x^2}\,dx = \int_0^4 \frac{1}{2}e^u\,du = \frac{1}{2}e^u\big|_0^4$

Thus, we find that

$$\int_0^2 xe^{x^2}\,dx = \frac{1}{2}(e^4 - 1).$$

Approach 2 works provided certain conditions on f and g are met:

$$\int_a^b f(g(x))\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

if

1. g' is continuous on $[a, b]$.
2. f is continuous on the set of values taken by g on $[a, b]$.

Substitutions are useful or necessary for a huge range of integrals. You will find yourself either implicitly or explicitly using a substitution in virtually every integral you compute!

Key Concepts

The substitution method amounts to applying the Chain Rule in reverse:

To compute $\int f(g(x))g'(x)\,dx$, we let $u = g(x)$ $du = g'(x)dx$.

Then $\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) = F(g(x))$ where F is an antiderivative of f.

[I'm ready to take the quiz.] [I need to review more.]
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