Harvey Mudd College Math Tutorial:

Multivariable Functions, Surfaces, and Contours

The graphs of surfaces in 3-space can get very intricate and complex! In this tutorial, we investigate some tools that can be used to help visualize the **graph** of a function $f(x, y)$, defined as the graph of the equation $z = f(x, y)$.

Try plotting

$z = \sin(xy)$!

Example

Let $f(x, y) = x^2 + \frac{y^2}{4}$. Before actually graphing $z = x^2 + \frac{y^2}{4}$, let's see if we can visualize the surface that will result.

If we set $y = 0$, we find that the intersection of the surface with the xz-plane is the parabola $z = x^2$.

Similarly, setting $x = 0$, the intersection of the surface with the yz-plane is the parabola $z = \frac{y^2}{4}$.

Can you see what the surface, called an elliptic paraboloid, will look like?

View Surface

By setting $x = 0$ or $y = 0$ in $z = f(x, y)$, we are really looking at the intersection of the surface $z = f(x, y)$ with the plane $x = 0$ or $y = 0$, respectively. If we take the intersection of a surface $z = f(x, y)$ with any plane, the resulting curve is called the **cross section** or **trace** of the surface in the plane.

Example

Let $f(x, y) = 5 - \sqrt{x^2 + y^2}$. What can we determine about the surface given by $z = 5 - \sqrt{x^2 + y^2}$? Notice that $z \leq 5$. If we set $z = 5$, $x^2 + y^2 = 0$ and we get a single point $x = 0, \quad y = 0$ in the plane $z = 5$.

If we set $z = 4$, $x^2 + y^2 = 1$, giving a circle of radius 1.

If $z = 0$, $x^2 + y^2 = 5$, a circle of radius $\sqrt{5}$.

If $z = -4$, $x^2 + y^2 = 9$, a circle of radius 3.

Is this another paraboloid? Notice that the trace in the plane $y = 0$ is the pair of lines $z = 5 - x$ and $z = 5 + x$.

Similarly, the trace in the plane $x = 0$ is the pair of lines $z = 5 - y$ and $z = 5 + y$. The surface is a right circular cone.

View Surface

When we take the intersection of the surface $z = f(x, y)$ with the horizontal plane $z = k$, as we did several times in the previous example, the projection of the resulting curve onto the xy-plane is called the **level curve of height** k. Along this curve, f is constant with value k.

A collection of level curves of a surface, labeled with their heights, is called a **contour map**.

A contour map is just a topographic map of the surface.

Example
Let $f(x, y) = \sqrt{9 - x^2 - y^2}$. Notice here that $f(x, y) \geq 0$. We will examine the level curves of $z = f(x, y)$.

Setting $z = k, \quad k \geq 0$, squaring both sides of the equation and rearranging terms, we find that the level curves of $z = f(x, y)$ are circles given by $x^2 + y^2 = 9 - k^2$.

Examination of traces with $x = c$ or $y = c$ shows them to be portions of circles. Thus, $z = f(x, y)$ is a hemisphere here.

Squaring $z = \sqrt{9 - x^2 - y^2}$ from the previous example and rearranging terms, we obtain $x^2 + y^2 + z^2 = 9$, the equation of a sphere. It is useful to be able to recognize some common quadric surfaces such as this.

Note

For a function $f(x, y, z)$ of *three* variables, $f(x, y, z) = k$ is called the **level surface with constant** k. The function $f(x, y, z)$ is constant over the level surface.

Key Concept

Let $z = f(x, y)$.

The projection onto the xy-plane of the intersection of the surface $z = f(x, y)$ with the horizontal plane $z = k$ is called the **level curve of height** k. A collection of level curves, called a **contour map** is a useful tool in visualizing the graph of a function $f(x, y)$.

[I'm ready to take the quiz.] [I need to review more.]
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