Harvey Mudd College Math Tutorial:

Tangent Planes and Linear Approximation

Just as we can visualize the line tangent to a curve at a point in 2-space, in 3-space we can picture the **plane** tangent to a **surface** at a point.

Consider the surface given by $z = f(x, y)$. Let (x_0, y_0, z_0) be any point on this surface. If $f(x, y)$ is **differentiable** at (x_0, y_0), then the surface has a **tangent plane** at (x_0, y_0, z_0). The equation of the tangent plane at (x_0, y_0, z_0) is given by

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

Notes

- Recall that the equation of the plane containing a point (x_0, y_0, z_0) and normal to the vector $\mathbf{n} = (a, b, c)$ is

  $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

  The **derivation** of the equation for the tangent plane just involves showing that the tangent plane is normal to the vector $\mathbf{n} = (f_x(x_0, y_0), f_y(x_0, y_0), -1)$.

- For surfaces $F(x, y, z) = 0$ that are not easily solved for z, the equation of the tangent plane at (x_0, y_0, z_0) is

  $$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

  provided that $\nabla F(x_0, y_0, z_0) \neq 0$. Note that if we let $F(x, y, z) = f(x, y) - z$, we obtain the equation given for the tangent plane to $z = f(x, y)$ at (x_0, y_0, z_0).

Example
Let's find the equation of the plane tangent to the surface $z = 4x^3y^2 + 2y$ at the point $(1, -2, 12)$.

Since $f(x, y) = 4x^3y^2 + 2y$,

$$f_x(x, y) = 12x^2y^2 \text{ and } f_y(x, y) = 8x^3y + 2.$$

With $x = 1$ and $y = -2$,

$$\begin{aligned} f_x(1, -2) &= 12(1)^2(-2)^2 = 48 \\ f_y(1, -2) &= 8(3)^3(-2) + 2 = -14. \end{aligned}$$

Thus, the tangent plane has normal vector $\mathbf{n} = (48, -14, -1)$ at $(1, -2, 12)$ and the equation of the tangent plane is given by

$$48(x-1) - 14(y-(-2)) - (z-12) = 0.$$

Simplifying,

$$48x - 14y - z = 64.$$

Linear Approximation

The tangent plane to a surface at a point stays close to the surface near the point. In fact, if $f(x, y)$ is differentiable at the point (x_0, y_0), the tangent plane to the surface $z = f(x, y)$ at (x_0, y_0) provides a good approximation to $f(x, y)$ near (x_0, y_0):

Solving $f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$ for z,

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Since $z_0 = f(x_0, y_0)$, we have that

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Near (x_0, y_0), the surface is close to the tangent plane. Thus,

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

We call this the **linear approximation** or **local linearization** of $f(x, y)$ near (x_0, y_0).

Notes

- The linear approximation is really just the multivariable Taylor polynomial of degree 1 for $f(x, y)$ about (x_0, y_0). It is only accurate near (x_0, y_0). Better approximations can be obtained by using higher-order Taylor polynomials.
- These concepts can be extended to functions of more than two variables:

  $$f(x, y, z) \approx f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0)$$

  where $f(x, y, z)$ is differentiable at (x_0, y_0, z_0).

Example
From our work in the previous example, the linear approximation to $f(x, y) = 4x^3y^2 + 2y$ near $x = 1, \quad y = -2$ is

$$f(x, y) \approx 48x - 14y - 64.$$

This is, of course, exact at $x = 1, \quad y = -2$:

$$f(1, -2) = 12 = 48(1) - 14(-2) - 64.$$

At $x = 1.1$ and $y = -1.9$, according to the linear approximation,

$$f(1.1, -1.9) \approx 48(1.1) - 14(-1.9) - 64 = 15.4,$$

which is indeed very close to the exact value $f(1.1, -1.9) = 15.41964$.

Key Concepts

- **Tangent Plane to a Surface**

  Let (x_0, y_0, z_0) be any point on the surface $z = f(x, y)$. If $f(x, y)$ is differentiable at (x_0, y_0), then the surface has a tangent plane at (x_0, y_0, z_0) given by

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

- **Linear Approximation to a Surface**

  If $f(x, y)$ is differentiable at (x_0, y_0), then near (x_0, y_0)

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

[I'm ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]