Harvey Mudd College Math Tutorial:

Functions and Transformations of Functions

We will review some of the important concepts dealing with functions and transformations of functions. Most likely you have encountered each of these ideas previously, but here we will tie the concepts together.

Definition of a Function

Let A and B be sets.

A **function** $F : A \to B$ is a relation that assigns to each $x \in A$ a unique $y \in B$. We write $y = f(x)$ and call y the **value of** f **at** x or the **image of** x **under** f. We also say that f **maps** x to y.

The set A is called the **domain** of f. The set of all possible values of $f(x)$ in B is called the **range** of f. Here, we will only consider real-valued functions of a real variable, so A and B will both be subsets of the real numbers. If A is left unspecified, we will assume it to be the largest set of real numbers such that for all $x \in A$, $f(x)$ is real.

Examples

Click on a bullet to see the graph of that function as well as its domain and range.

- $f(x) = x^2$, x real.
- $f(x) = \sin x$, x real.
- $f(x) = \begin{cases} -1, & x < 0 \\ 2, & x \geq 0 \end{cases}$
- $f(x) = \dfrac{1}{x+1}$.
- $f(x) = \sqrt{x-3}$.

Even/Odd Functions

A function $f : A \to B$ is said to be **even** if and only if

$$f(-x) = f(x) \quad \text{for all } x \in A$$

and is said to be **odd** if and only if

$$f(-x) = -f(x) \quad \text{for all } x \in A.$$

Most functions are neither even nor odd.

The graph of an even function is **symmetric about the y-axis**, while the graph of an odd function is **symmetric about the origin**.

Of the functions in the example,

- $f(x) = x^2$ is even.
- $f(x) = \sin x$ is odd.
- The others are neither even nor odd.

Transformations of Functions

We will examine four classes of transformations, each applied to the function $f(x) = \sin x$ in the graphing examples.

- **Horizontal translation:** $g(x) = f(x + c)$.
  The graph is translated c units to the *left* if $c > 0$ and c units to the *right* if $c < 0$.
  Graph
- **Vertical translation:** $g(x) = f(x) + k$.
  The graph is translated k units *upward* if $k > 0$ and k units *downward* if $k < 0$.
  Graph
- **Change of amplitude:** $g(x) = Af(x)$.
  The amplitude of the graph is increased by a factor of A if $|A| > 1$ and decreased by a factor of A if $|A| < 1$. In addition, if $A < 0$ the graph is inverted.
  Graph
- **Change of scale:** $g(x) = f(ax)$.
  The graph is "compressed" if $|a| > 1$ and "stretched out" if $|a| < 1$. In addition, if $a < 0$ the graph is reflected about the y-axis.
  Graph

In the Exploration, experiment with applying several transformations to a single function.

Exploration

Key Concepts

A **function** F: $A \longrightarrow B$ is a relation that assigns to each $x \in A$ a unique $y \in B$. We write $y = f(x)$ and call y the **value of f at x** or the **image of x under f**. We also say that f **maps** x to y.

The set A is called the **domain** of f. The set of all possible values of $f(x)$ in B is called the **range** of f.

Each of these transformations takes a function f and produces a new function g:

- **Horizontal translation:** $g(x) = f(x + c)$.
  The graph is translated c units to the *left* if $c > 0$ and c units to the *right* if $c < 0$.

- **Vertical translation:** $g(x) = f(x) + k$.
  The graph is translated k units *upward* if $k > 0$ and k units *downward* if $k < 0$.

- **Change of amplitude:** $g(x) = Af(x)$.
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[I'm ready to take the quiz.] [I need to review more.]
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