Due: Wednesday, April 21

HMC Math 143 Spring 2004

Prof. Gu

Problem Set 9

Start this assignment before Sunday night!

Read:

- Handout on Integration on Manifolds.
- Lecture Notes.

Do:

A: Problems on Reviewing Volume Form.

- a) Show that $dx_1 \wedge dx_2 \wedge ... \wedge dx_n(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}) = 1$.
- b) Write in your own words to explain why $dV = \sqrt{det(g_{ij})}dx_1 \wedge dx_2 \wedge ... \wedge dx_n$. Moreover, show that dV does not depend on the choice of local coordinates.

B: Problems from Lectures

- a) Show that Stokes's theorem is true over the standard 3-cube $I^3 : [0,1]^3 \to R^3$ defined by $I^3(x) = x$ for $x \in [0,1]^3$.
- b) Show that the classic Stokes' theorem on the last page of the Handout is a special case of the general Stokes's theorem of a manifold with boundry.

C: Other Problems See next page.

1. If M is a k-dimensional manifold-with-boundary, prove that ∂M is a $(k-1)$-dimensional manifold and $M - \partial M$ is a k-dimensional manifold.

2. (a) Let $A \subset \mathbf{R}^n$ be an open set such that boundary A is an $(n-1)$-dimensional manifold. Show that $N = A \cup \text{boundary}\, A$ is an n-dimensional manifold-with-boundary. (Bear in mind the following example: if $A = \{x \in \mathbf{R}^n : |x| < 1 \,\text{or}\, 1 < |x| < 2\}$ then $N = A \cup \text{boundary}\, A$ is a manifold-with-boundary, but $\partial N \neq \text{boundary}\, A$.)

   (b) Prove a similar assertion for an open subset of an n-dimensional manifold.

3. If ω is a $(k-1)$-form on a compact k-dimensional manifold M, prove that $\int_M d\omega = 0$. Give a counterexample if M is not compact.

4. If $M_1 \subset \mathbf{R}^n$ is an n-dimensional manifold-with-boundary and $M_2 \subset M_1 - \partial M_1$ is an n-dimensional manifold-with-boundary, and M_1, M_2 are compact, prove that

$$\int_{\partial M_1} \omega = \int_{\partial M_2} \omega,$$

   where ω is an $(n-1)$-form on M_1, and ∂M_1 and ∂M_2 have the orientations induced by the usual orientations of M_1 and M_2. *Hint:* Find a manifold-with-boundary M such that $\partial M = \partial M_1 \cup \partial M_2$ and such that the induced orientation on ∂M agrees with that for ∂M_1 on ∂M_1 and is the negative of that for ∂M_2 on ∂M_2.

5. Generalize the divergence theorem to the case of an n-manifold with boundary in $\mathbf{R}^n$.

6. Applying the generalized divergence theorem to the set $M = \{x \in \mathbf{R}^n : |x| \leq a\}$ and $F(x) = x$, find the volume of $S^{n-1} = \{x \in \mathbf{R}^n : |x| = 1\}$ in terms of the n-dimensional volume of $B_n = \{x \in \mathbf{R}^n : |x| \leq 1\}$. The volume of B_n is given by

$$\text{Vol}(B_n) = \begin{cases} \frac{\pi^{n/2}}{(n/2)!} & n \text{ even} \\ \frac{2^{(n+1)/2}\pi^{(n-1)/2}}{1\cdot 3\cdot 5\cdot \cdots \cdot n} & n \text{ odd.} \end{cases}$$