

Math 136: Complex Variables
Fall 2002
Midterm Exam 1

October 2, 2002

Name: _____

Date: _____ Start Time: _____ am/pm

Date: _____ Finish Time: _____ am/pm

Instructions: This is a closed book, closed notes, 3 hour take home exam. The three hours must be in one continuous block of time. The HMC honour code applies to the taking of this exam. You may use the one page of notes you have prepared. You are not allowed to consult any other notes, or printed source, or people, during the exam, **except:** If you have any questions about the exam, contact Dr. Ward (x76019 or ward@math.hmc.edu).

Justify your answers, and show your work. Partial credit will be given. You may use the results of earlier problems or parts of problems, even if you have not proved them. The exam is due in class at 2:45pm on **Monday, 7 October**. I will be happy to accept it earlier; just put it under my office door. If you need an extension, see me as soon as possible.

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Bonus	4	
Total	100	

1. (20 points) Short answer question. No proofs required.

(a) (3 points) State Cauchy's Theorem (i.e. Cauchy's Integral Theorem, not the Cauchy Integral Formula).

(b) (4 points) Find and plot the cube roots of $z_0 = -\sqrt{2} + \sqrt{2}i$.

(c) (4 points) To what region does the function $f(z) = e^z$ map the unit square S whose corners are at 0 , 1 , i , and $1 + i$? Sketch the region $f(S)$.

(d) (3 points) Find the principal value of $(-2i)^i$.

(e) (4 points) Let Γ be the positively oriented contour consisting of the line segment from $1 - 4i$ to $1 + 4i$, together with the left half of the circle of centre 1 and radius 4 . Sketch Γ and give the value of the integral

$$\int_{\Gamma} \left[\frac{4}{z-2} + \frac{3}{z-2i} + \frac{2i}{(z+2)^2} - 5(z+2i) \right] dz.$$

(f) (2 points) What's your favourite complex analysis fact, so far?

2. (10 points) Use the limit definition of the derivative to prove that:

(a) The function $f(z) = \operatorname{Re}(z)$ is nowhere differentiable.

(b) Every real-valued analytic function $f : \mathbb{C} \rightarrow \mathbb{R}$ is constant.

3. (10 points) (a) Let γ be the line segment from the origin to the point $5i$. Compute the contour integral

$$\int_{\gamma} \frac{\bar{z}}{1-z^2} dz.$$

- (b) Sketch the arc γ_2 from $-1-i$ to $1+i$ of the curve $y = x^5$. Write down (but don't evaluate!) an integral with respect to t for the integral over γ_2 of the function in part (a).

4. (10 points) Let $u(x, y) = x^3 - kxy^2$.

(a) For which value(s) of k is u harmonic?

(b) For a value of k you found in part (a), find a harmonic conjugate function for u .

(c) For the same value of k , find an analytic function $f(z)$ whose real part is u . Write $f(z)$ explicitly as a function of z .

5. (10 points) Let Γ be the boundary, traversed counterclockwise, of the triangle with vertices at the origin, $3i$, and -4 . Sketch Γ . Show that

$$\left| \int_{\Gamma} (e^z - \bar{z}) dz \right| \leq 60.$$

Bonus: (1 point) Can you do better (smaller) than 60?

6. (10 points) (a) Derive a formula for the inverse tangent function

$$\tan^{-1} z.$$

(b) Use your formula to find the value(s) of $\tan^{-1}(1)$. Plot answer(s) on the complex plane.

7. (10 points) Let C be the circle of radius 2 centred at the origin, positively oriented. Evaluate the integral around C of the function

$$f(z) = z^{-1+3i},$$

using the branch of z^{-1+3i} such that $0 < \arg(z) \leq 2\pi$.

8. (10 points) Consider the function $f(z) = \log(z + \frac{1}{z})$ defined using the branch of $\log w$ such that $\pi/2 < \arg w \leq 5\pi/2$. For which z is $f(z)$ continuous? Justify your answer.

9. (10 points) The *Maximum Modulus Principle* for analytic functions states:

Theorem: If $f(z)$ is analytic and non-constant in a domain D , then its absolute value $|f(z)|$ has no maximum in D .

Prove the Maximum Modulus Principle, under the extra assumption that f' is never 0 in D . *Hint:* Write $f = u + iv$, and consider the gradients and level curves of u and v at a point z_0 in D .

Bonus: (3 points) Prove that the equation $e^z = z$ has infinitely many solutions.