Home Work Solution Set #3

Math 14: Gupta, Limon

1. Problem 1: Find The Limits

   (a) Find $\lim_{(x,y)\to(0,0)} \ln\sqrt{1-x^2-y^2}$
   Solution: Note that the square root and natural log functions are both continuous, thus we can simply plug in $(x=0, y=0)$. We get $\ln\sqrt{1-0-0}=0$, so the limit is zero.

   (b) Find $\lim_{(x,y)\to(0,0)} \frac{e^{xy}\sin xy}{xy}$
   Solution: Note that the term xy appears multiple times in the equation, so let $u=xy$. Thus, $\lim_{u\to 0}\frac{e^u \sin u}{u} = 1*1 = 1$ (remember your series expansions, $\sin(u)/u$ has a removable singularity).

   (c) Find $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$
   Solution: Note the form of the denominator, so transform the function to polar coordinates. Then $\lim_{r\to 0}\frac{r^3(\cos^3\theta-\sin^3\theta)}{r^2}$, and after factoring, we get $\lim_{r\to 0} r(\cos^3\theta-\sin^3\theta)=0$ (via squeeze theorem or linear vs. bounded argument).

   (d) Find $\lim_{(x,y)\to(0,0)} \frac{e^{x^2+y^2}-1-x^2-y^2}{(x^2+y^2)^2}$
   Solution: Note that the term x^2+y^2 appears multiple times, so let $u=x^2+y^2$. Thus $\lim_{u\to 0}\frac{e^u-1-u}{u^2}$, and after applying L'Hopital's rule once, we get $\lim_{u\to 0}\frac{e^u-1}{2u}$. We applying it a second time and get $\lim_{u\to 0}\frac{e^u}{2}=\frac{1}{2}$.

   (e) Find $\lim_{(x,y)\to(0,0)} \frac{x^{10}+y^{10}}{x^2+y^2}$
   Solution: Factor out a common x^2+y^2 term,
   $\lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(x^8-x^6y^2+x^4y^4-x^2y^6+y^8)}{(x^2+y^2)} = 0$.

   (f) Find $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$
   Solution: Transform to polar coordinates, so $\lim_{r\to 0}\frac{(r^2\cos^2\theta)(r\sin\theta)}{r^2}$, and after factoring, $\lim_{r\to 0} r\cos^2\theta\sin\theta = 0$ (via squeeze theorem or linear vs. bounded argument).

2. Problem 2: Do The Limits Exist?

   (a) Does $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ exist?
   Solution: Remember that for a counter example, any two paths resulting in different limits will work. So take the $y = x$ path, thus resulting in $\lim_{x\to 0} \frac{0}{2x^2} = 0$, and then take the path $y = 2x$, which results in $\lim_{x\to 0} \frac{x^2-4x^2}{x^2+4x^2} = \lim_{x\to 0} \frac{-3x^2}{5x^2} = \frac{-3}{5}$. Because two different paths result in two different limits, the limit does not exist.

   (b) Does $\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$ exist?
   Solution: Take the path $y = x$, so that $\lim_{x\to 0} \frac{x^3x}{x^6+x^2}$. After factoring and simplifying, we get $\lim_{x\to 0} \frac{x^4}{x^2(x^4+1)} = \lim_{x\to 0} \frac{x^2}{x^4+1} = 0$. Now use the path $y = x^3$, so that $\lim_{x\to 0} \frac{x^3x^3}{x^6+x^6}$. After factoring and simplifying, we get $\lim_{x\to 0} \frac{x^6}{2x^6} = \frac{1}{2}$. Because two different paths result in two different limits, the limit does not exist.