MATH 14 MIDTERM REVIEW SOLUTIONS

WARNING: The following problems do not cover all the material covered in class. You MUST go over your notes and old homework in order to obtain a complete review.

1. Circle **T** for true, or **F** for false for each question. You do NOT have to show your work.

   T F (a) If a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous at $\vec{x} = \vec{a}$, then f must have a limit as $\vec{x}$ approaches $\vec{a}$.

   Solution: True by the definition of continuity.

   T F (b) If $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is differentiable at $\vec{x} = \vec{a}$, then both partial derivatives, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ exist at $\vec{x} = \vec{a}$.

   Solution: True since by the definition of differentiability.

   T F (c) If $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, and both partials, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ exist at $\vec{x} = \vec{a}$, then f is differentiable at $\vec{x} = \vec{a}$.

   Solution: False since existence of the partial derivatives is not enough to guarantee differentiability.

   T F (d) Suppose $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a function. If the level curve for $c = 1$ is the line $x = 1$ (minus the point (1,1)) and the level curve for $c = 2$ is the line $y = x$ (minus the point (1,1)), then f has no limit as $(x, y) \longrightarrow (1, 1)$.

   Solution: True since then there would be 2 different paths with 2 different limits.

2. Find an equation of the plane that passes through the points $(1, 5, 1), (3, 1, 3)$, and $(-3, 4, 3)$. Your answer should be in the form $Ax + By + Cz = D$.

Solution: The method is outlined in the notes handed out the first day of class. The desired plane is $x+2y+3z=14$.

3. Compute each limit or show that the limit does not exist:

   (a) $\lim_{(x,y)\to(0,0)} 14e^{-x^2-y^2}$
   Solution: Plug in. The limit is 0.

   (b) $\lim_{(x,y)\to(0,0)}(\frac{x^{12}-y^{12}}{x^4+y^4}+14)$
   Solution: Use fancy factoring. The limit is $0+14=14$.

   (c) $\lim_{(x,y)\to(0,0)} \frac{84e^{x^2+y^2}-84-84x^2-84y^2-42(x^2+y^2)^2}{(x^2+y^2)^3}$
   Solution: Let $u=x^2+y^2$, and repeatedly use l'hopital's rule. The desired limit is 14.

   (d) $\lim_{(x,y)\to(0,0)}(x\sin(\frac{x}{x^2+y^2})+14)$
   Solution: Use polar coordinates followed by the squeeze theorem. The desired limit is 14.

4. Find an equation of the tangent plane to the surface at the specified point.

   (a) $z=x^{14}+y^{14}$ at the point $(1,1,2)$.
   Solution: We use the equation of the tangent plane:

   $$z=f(a,b)+\nabla f(a,b)\cdot(x-a,y-b).$$

   We get $z=2+14(x-1)+14(y-1)$.

   (b) $x^{14}+y^{14}+z^{14}=3$ at the point $(1,1,1)$.
   Solution: We use the equation of the tangent plane

   $$\nabla w(a,b,c)\cdot(x-a,y-b,z-c)=0,$$

   where $w=x^{14}+y^{14}+z^{14}-3$. We get $14(x-1)+14(y-1)+14(z-1)=0$.

5. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, f(x,y) = (x^{14}y^{14}, \frac{1}{2}x^{28}y^{28}, 14x + 14y)$. Compute $Df(1,1)$.

   Solution: The derivative matrix is 3×2, and is

$$\begin{bmatrix} 14 & 14 \\ 14 & 14 \\ 14 & 14 \end{bmatrix}.$$

6. Compute the directional derivative of the function $T(x,y) = x^3 - y^2$ at the point $(1,2)$ in the direction $(5,12)$.

   Solution: Compute $\nabla f(1,2) = (3,-4)$. Let $\mathbf{u} = \frac{(5,12)}{13}$ be the direction vector. Hence $Df_u = (3,-4) \cdot \frac{(5,12)}{13} = \frac{-33}{13}$.

7. A function $f(x,y)$ is said to be harmonic if it satisfies Laplace's equation: $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$. Two functions u and v are said to satisfy the Cauchy-Riemann equations if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Prove that if u and v satisfy the Cauchy-Riemann equations, then u and v are harmonic.

   Solution: We have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, hence $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$. Also $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ implies $\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$. Adding the two equations we get $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$, since the continuity of the derivatives implies continuity of the mixed partials. v is harmonic by the same technique.