

**Zoo Guide: Discrete Distributions I**

This is the first of four installments that will provide a guide to the “zoo” of probability distributions. Its goal is to provide a concise summary of the key facts for the most important probability distributions, and indications of the relationships among them. (Most of this information appears in the book, but is scattered among the text and exercises of various chapters.)

The distributions described in this first installment all relate, directly or as limiting cases, to a sequence of independent Bernoulli trials that succeed with probability  $p$  (and thus fail with probability  $1 - p$ ). More concretely, one may think of a sequence of independent flips of a biased coin that comes up heads with probability  $p$  (and thus comes up tails with probability  $1 - p$ ).

*Bernoulli:*  $X \sim \text{Bernoulli}(p)$  means  $X \in \{0, 1\}$ ,  $\Pr(X = 1) = p$  and  $\Pr(X = 0) = 1 - p$ . We have  $\text{Ex}[X] = p$ ,  $\text{Var}[X] = p(1 - p)$  and the probability generating function  $g_X(t) = \text{Ex}[t^X] = pt + 1 - p$ . The distribution is named in honor of Jacob (Jacques) Bernoulli (1654-1705).

*Binomial:*  $X \sim \text{Binomial}(n, p)$  means  $X \in \{0, 1, \dots, n\}$  and  $\Pr(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$  for  $0 \leq k \leq n$ . We have  $\text{Ex}[X] = np$ ,  $\text{Var}[X] = np(1 - p)$  and the probability generating function  $g_X(t) = (pt + 1 - p)^n$ .  $X$  has the same distribution as the sum of  $n$  independent Bernoulli( $p$ ) random variables. The distribution gets its name from the form of its generating function.

*Geometric:*  $X \sim \text{Geometric}(p)$  means  $X \in \{1, 2, \dots\}$  and  $\Pr(X = k) = p(1 - p)^{k-1}$  for  $k \geq 1$ . Some authors subtract 1 to get a random variable  $Y$ , with a distribution we shall designate  $\text{Geometric}^*(p)$ , such that  $Y \in \{0, 1, \dots\}$  and  $\Pr(Y = k) = p(1 - p)^k$  for  $k \geq 0$ . We have  $\text{Ex}[X] = 1/p$ ,  $\text{Var}[X] = (1 - p)/p^2$  and the probability generating function  $g_X(t) = pt/(1 - (1 - p)t)$ .  $X$  (respectively,  $Y$ ) has the distribution of the number of trials (respectively, failures) needed to get the first success in a sequence of independent Bernoulli( $p$ ) trials. The distribution gets its name from the progression of its probabilities.

*Negative Binomial:*  $X \sim \text{NegBinomial}(k, p)$  means  $X \in \{k, k + 1, \dots\}$  and  $\Pr(X = n) = \binom{n-1}{k-1} p^k (1 - p)^{n-k}$  for  $n \geq k$ . Some authors subtract  $k$  to get a random variable  $Y$ , with a distribution we shall designate  $\text{NegBinomial}^*(k, p)$ , such that  $Y \in \{0, 1, \dots\}$  and  $\Pr(Y = n) = \binom{n+k-1}{k-1} p^k (1 - p)^n$ . We have  $\text{Ex}[X] = k/p$ ,  $\text{Var}[X] = k(1 - p)/p^2$  and the probability generating function  $g_X(t) = (pt/(1 - (1 - p)t))^k$ .  $X$  (respectively,  $Y$ ) has the distribution of the number of trials (respectively, failures) needed to get the  $k$  successes in a sequence of independent Bernoulli( $p$ ) trials. It is the same as the distribution of the sum of  $k$  independent Geometric( $p$ ) (respectively,  $\text{Geometric}_0(p)$ ) random variables. The distribution gets its name from the form of its generating function.

*Poisson:*  $X \sim \text{Poisson}(\lambda)$  means  $X \in \{0, 1, \dots\}$  and  $\Pr(X = k) = e^{-\lambda} \lambda^k / k!$  for  $k \geq 0$ . We have  $\text{Ex}[X] = \lambda$ ,  $\text{Var}[X] = \lambda$  and the probability generating function  $g_X(t) = e^{\lambda(t-1)}$ .  $X$  has the limiting distribution of a sequence of Binomial( $n, \lambda/n$ ) random variables as  $n \rightarrow \infty$ , which is the same as the limiting distribution of a sequence of NegBinomial $^*(k, 1 - \lambda/k)$  random variables as  $k \rightarrow \infty$ . The distribution is named in honor of Siméon-Denis Poisson (1781-1840).