

## Math 14 Multivariable Calculus Sample Exam Questions I

1. Find the projection of the vector  $\mathbf{i} + \mathbf{j} + \mathbf{k}$  on the vector  $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ .
2. (10) Using vector methods, prove that the diagonals of a parallelogram bisect each other.
3. (10) Using vector methods, prove that the diagonals of a parallelogram are perpendicular if, and only if, the parallelogram is a rhombus.
4. (10) Analyze the following limit:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^4 + y^2}.$$

If the limit exists, identify the limit; if the limit does not exist, explain why not.

5. (15) Find the tangent plane and the normal line to the surface  $x^2 + y^2 + z^2 = 9$  at the point  $(2, 2, 1)$ .
6. (10) Find the directional derivative of the function  $f(x, y, z) = 2x^2 - y^2 + z^2$  at  $(1, 2, 3)$  in the direction from  $(1, 2, 3)$  to  $(3, 5, 0)$ .
7. (10) Define two functions by:

$$f(x, y) = (xy, x + y, x/y), \quad g(u, v, w) = (uvw, u + v + w).$$

What is  $Dg \circ f(1, 1)$ ?

8. (15) Show that the curve defined by the equations

$$x^2 - y^2 + z^2 = 1, \quad xy + xz = 2,$$

is tangent to the surface

$$xyz - x^2 - 6y = -6$$

at the point  $(1, 1, 1)$ .

9. (10) Define the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

- (a) Compute  $\frac{\partial f}{\partial x}(0,0)$  and  $\frac{\partial f}{\partial y}(0,0)$ .
- (b) Compute  $\nabla f(0,0)$ .
- (c) Compute the directional derivative at  $(0,0)$  in the direction of  $\mathbf{i} + \mathbf{j}$ . Justify your calculation.
10. (15) Consider a point  $P$  on the rim of a wheel of radius  $r$ . This wheel rolls without slipping on the outside of a wheel of radius  $R$ . Find parametric equations of the path traced out by the point  $P$ . This path is called an *epicycloid*.
11. (15) Let  $\mathbf{c}(t)$  represent a differentiable path. Let  $\mathbf{v}(t)$  represent the velocity vector to the path at time  $t$ . The unit tangent vector to the path at time  $t$  is

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}.$$

- (a) Show that the vector  $T'(t) = \frac{d\mathbf{T}}{dt}$  is orthogonal to  $\mathbf{T}$ .
- (b) For a given  $t$ , the plane determined by the velocity  $\mathbf{v}$  and the acceleration  $\mathbf{a} = \mathbf{v}'$  is called the *osculating plane* to the path at  $\mathbf{c}(t)$ . Find an equation for the osculating plane to the helix

$$\mathbf{c} = (\cos t, \sin t, t)$$

for  $t = \pi$ .

12. (10) A function  $f(x, y)$  is said to be *homogeneous of degree  $n$*  if (for fixed  $n$ ),

$$f(tx, ty) = t^n f(x, y).$$

For example, the function  $f(x, y) = x^2 + y^2$  is homogeneous of degree  $n = 2$  because

$$f(tx, ty) = (tx)^2 + (ty)^2 = t^2(x^2 + y^2) = t^2 f(x, y).$$

Prove that a (differentiable) homogeneous function of degree  $n$  must satisfy the partial differential equation:

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f(x, y).$$