Math 14 Multivariable Calculus Sample Exam Questions II

1. Maisie the Wonder Dog has a very good nose for the neighbors BBQ. When Lisa is cooking steaks on the BBQ, the wonderful aroma has a concentration as a function of position that follows the formula

$$C(x,y) = e^{-2(x-10)^2-(y-5)^2}$$

   Maisie is at position $(1,1)$. If she starts to walk directly toward position $(10,5)$, what is the instantaneous rate of change of the concentration of the aroma?

   In what direction should Maisie move, from $(0,0)$ in order to maximize the rate at which the concentration of the aroma is increasing?

2. (15) Evaluate some integrals:

   (a) Integrate the function $f(x,y) = xy$ over that region of the plane between the two curves $\phi(x) = x^2$ and $\psi(x) = (1-x)^2$.

   (b) Integrate the function $f(x,y,z) = x+y-z$ over the solid region of space with $0 \le x, 0 \le y, 0 \le z$ that is underneath the plane $x+y+z=1$

   (c) Evaluate the integral $\int_0^1 \int_y^1 ye^{x^3} dx dy$.

3. (15) Let $\boldsymbol{F} = x^2\boldsymbol{i} - yz\boldsymbol{j} + (y+z)\boldsymbol{k}$.

   (a) Compute $\nabla \cdot \boldsymbol{F}$.

   (b) Compute $\nabla \times \boldsymbol{F}$.

   (c) Let $\boldsymbol{G}$ be a vector field and let f be a scalar function. Demonstrate the identity:

$$\nabla \times (f\boldsymbol{G}) = f(\nabla \times \boldsymbol{G}) + (\nabla f) \times \boldsymbol{G}.$$

4. Let $\mathcal{C}$ represent the curve that is the unit circle $x^2+y^2=1$. Determine a one-to-one parameterization of this curve that is oriented in the clockwise direction. Then evaluate the line integral:

$$\int_{\mathcal{C}} \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$$

5. Let the path $\mathcal{C}$ start at the point $(0,0,0)$ and proceeds in straight line segments to the point $(1,0,0)$, and then to the point $(1,1,0)$ and finally to the point $(1,1,1)$. Evaluate the path integral $\int_{\mathcal{C}} f\, ds$ for the scalar function $f(x,y,z) = x^2+y^2+z^2$.

6. (15) Consider the vector field $\boldsymbol{F}(x, y, z) = 2x\boldsymbol{i} + zy\boldsymbol{k}$. Let S be the portion of the plane $z = 2$ above the unit square in the $x - y$ plane (*i.e.,* $0 \leq x, y \leq 1$). Compute the flux integral $\int_S \boldsymbol{F} \cdot d\boldsymbol{S}$ where the surface S is oriented *downward* relative to the $+z$ direction.

7. (15) Let S represent the lateral side of the cylinder of radius 2 and height 3 whose base is the disk $x^2 + y^2 \leq 4^2$ in the x-y plane. Define the vector field $\boldsymbol{F}$ in the following way: At position (r, θ, z) *in cylindrical coordinates*, $\boldsymbol{F}(r, \theta, z) = \theta \cos\theta \boldsymbol{i} + \theta \sin\theta \boldsymbol{j}$. Evaluate $\int_S \boldsymbol{F} \cdot d\boldsymbol{S}$ where the cylinder S is oriented *outward*.

8. (15) Let S represent the sphere of radius 1 centered at the origin, oriented outward, and define the vector field field $\boldsymbol{F}(x, y,) = z^2\boldsymbol{k}$. Evaluate

$$\int_S \boldsymbol{F} \cdot d\boldsymbol{S}.$$