Math 62 Probability Sample Exam Questions

Part 1: Short Calculations

1. A certain grand jury in Los Angeles has 30 members. If the jurors are selected at random from the adult population, of which 55% are female, what is the mean and standard deviation of the number of women jurors?

2. The uniform distribution over the interval $[0, L]$ has the cumulative probability distribution function given by

$$\Phi(x) = \begin{cases} 0, & x \leq 0 \\ x/L, & 0 < x \leq L \\ 1, & L < x \end{cases}$$

Calculate the mean and variance of the uniform distribution.

3. What is the probability that, if 200 fair coins are tossed, exactly 100 heads will be observed? Using *Stirling's approximation*,

$$n! \approx \sqrt{2\pi n} n^n e^{-n},$$

to estimate your answer numerically.

4. Given that a family of three children has at least one girl, what is the probability that it has exactly three girls?

5. Consider the following gambling game, called the Petersberg Game: Toss a fair repeatedly until the first tail is observed. If the tail occurs on the first toss, you do not win anything. If the tail occurs on toss $k+1$, you win 2^{k+1} dollars. Compute the average duration of the game, and compute the expected winnings of the player. Are you surprised by your results?

6. In a standard deck of 52 cards there are four suits: hearts $\heartsuit$, diamonds $\diamondsuit$, clubs $\clubsuit$ and spades $\spadesuit$, with thirteen cards per suit. In a standard game of poker, a hand of five cards is called a *flush* if all these cards are of the same suit. What is the probability that five cards selected at random from a standard deck form a flush?

7. On a recent trip to Boston, I visited my friend Andrew and he took me to breakfast at a classic 1950's diner. As we were sitting at a booth, I noticed that at the counter there were 11 barstools, and that six people were seated there. What intrigued me was that the none of the six people were seated immediately adjacent to anyone else. On the back of a napkin I did a quick calculation and convinced myself that it was highly unlikely that the people at the counter seated themselves randomly. What do you think? Support your conclusion with an appropriate calculation. Feel free to attach to this exam any (clean!) napkins that you used for scratch work.

8. Larry Bird's lifetime free-throw percentage was about 90%; we will take this statistic to mean that, on any given free throw, Larry the Legend had a chance of $p = 0.1$ of missing the shot, and a chance of $q = 0.9$ of making it. Assume that Larry took 12 free throws per game (which was his average). In what fraction of Larry's large number of games would you predict that he made 10 or more free throws?

9. Three different Airlines, called AM, UN and SW, fly out of Ontario. AM airline has 70 flights per day, of which 10% are late departures. UN airline has 50 flights per day, of which 8% are late, and SW has 65 flights per day, of which 13% are late. You randomly hear someone at the airport complaining about their late flight, but do not hear them say which airline. What is the probability that they were travelling on SW airline?

10. On standardized IQ tests, the results are normalized so that the average IQ score is 100 and the standard deviation is 15, and follows very closely a normal distribution. The average IQ of graduate students at CalTech is about 139. What fraction of the population has an IQ at least this large?

Part II

1. My office has a variety of coffee cups; normally when I use a cup, I wash it at the end of the day. Every now and then I forget to do so, and sometimes it results in an interesting mycological experiment. Recently I reached for a cup right before class, and noted that I had left a bit of sugared coffee in it a few days before, and the residue was a fertile medium for some sort of mold. After my initial reaction of disgust, I noticed that there was an interesting pattern to the colonies of mold in the cup. So I took a picture of the cup with my Sony Mavica digital camera; shown below is a version of that image that I modified using a graphics program so that it would reproduce on the page better.

The small circles represent the colonies that were growing in the cup. I have superimposed a rectangular grid on the image. As you can see, the colonies are not evenly distributed in the cup.

Explain why the data for the number of colonies growing in a grid cell might reasonably be expected to follow a Poisson distribution. Analyze the data from the figure and fill in the empty spaces of the table giving the expected numbers of cells with k colonies on the assumption that the distribution is exactly Poisson with a mean number of colonies per cell that you determine from the data. Comment on how well your numbers compare.

k	N_k	Expected number
0		
1		
2		
3		
≥ 4		

2. In clinical nursing care facilities, there are about twice as many women as men, since the average lifespan of women is considerably longer than for men. A medical research project is underway to study geriatric problems. If a group of 100 patients is selected at random for the study, what would be the probability that 75 or more of the group would be female? What would be the smallest group of patients that should be selected in order to make the chance less that 0.05 of having more than 70% of them female?

3. Suppose that each birth of a baby is equally likely to be a male or a female.

(a) Given that a family has exactly n children, what is the probability that exactly k of them are female?

(b) As you know from experience, not all families have the same number of children. A good approximation to the distribution of family size is the Poisson distribution, with mean 2.5. Thus the probability that a family will have exactly n children is $(2.5)^n e^{-2.5}/n!$. What fraction of families will have exactly two boys and two girls?

(c) What is the probability that a family will have children of only one gender?

4. Five letters are written to five different people and five envelopes are appropriately addressed. If each letter is placed into a randomly selected envelope (one letter per envelope), compute the probability P_1 that at least one person receives the correct letter. How would your answer change if 100 letters were written?

Figure 1: HPD Moody's coffee cup colonies.