Harvey Mudd College
Mathematics

Senior Thesis
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Research Proposal: Fréchet convergence of planar domains and convergence of harmonic measure distribution functions

Ariel E. Barton

Faculty Advisor: Prof. Lesley Ward

1 Introduction

For a connected region Ω in the complex plane and a point $z_0 \in \Omega$, we can define the function $h_\Omega(r)$ as the probability that a particle that moves by Brownian motion, released at z_0, first hits the boundary of Ω somewhere inside a circle centered at z_0 with radius r. This function $h_\Omega(r)$ is called the harmonic measure distribution function of Ω with respect to z_0. Harmonic measure distribution functions have been studied in [6] and [7].

For any such Ω, we can find an appropriate $h_\Omega(r)$. The question then becomes, given an $h(r)$, can we construct an Ω such that $h_\Omega(r) = h(r)$?

2 Proposed Research

I would like to investigate the convergence of sequences of these functions h_{Ω_n}. That is, given a sequence of domains Ω_n such that their harmonic measure distribution functions $h_{\Omega_n}(r)$ converge to some function $h(r)$, I would like to find sufficient conditions for the domains Ω_n to converge to some region Ω. This would continue prior research, which found sufficient conditions for the functions h_{Ω_n} to converge given that the domains Ω_n converge.

I plan to read the relevant parts of [1], [2], and [5] to learn about harmonic measure, and [3], [4], [6], and [7] to learn the theory of harmonic measure distribution functions.

3 Prior Research

Prof. Ward and Marie Snipes have investigated the question of finding domains Ω such that $h_\Omega(r) = h(r)$ for a given $h(r)$ by finding a way to construct an appropriate domain Ω when $h(r)$ is a step function. One can approximate a more general function $h(r)$ by a convergent series of step functions $h_{\Omega_n}(r)$. The question then becomes whether the corresponding domains Ω_n converge to some region Ω, and if so, whether $h_\Omega(r) = h(r)$.

In [3], M. Snipes proved a sufficient condition for convergence of the functions h_{Ω_n} to h_Ω for simply-connected domains Ω_n and Ω. Namely, she found that it suffices for the Ω_n to converge to Ω in the sense of Fréchet; or, equivalently, the Riemann maps for the domains Ω_n converge to that for Ω uniformly on the closed unit disc. I want to find out if the condition she found is also necessary, which is tantamount to showing that the convergence of the functions h_{Ω_n} is sufficient for convergence of the domains Ω_n.

References

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[3] M. Snipes, HMC Senior Thesis, 1999

[4] O. Cortez, HMC Senior Thesis, 2000

[5] M. Tsuji, *Potential Theory in Modern Function Theory.* Chelsea Publishing Co., New York, 1975

[6] B.L. Walden and L.A. Ward, *Distributions of harmonic measure for planar domains.* Proceedings of the 16th Nevanlinna Colloquium, Joensuu, Finland, 1995. Eds. Laine and Martio, Walter de Gruyter & Co. (1995), 289 - 299. (UNSW Pure Mathematics Report PM95/21.)

[7] B.L. Walden and L.A. Ward, *Asymptotic behaviour of distributions of harmonic measure for planar domains.* Complex Variables: Theory and Applications, Vol. 46 (2001), 157 - 177.