On the Erdős Problem of Empty Convex Hexagons

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December, 2003

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Abstract

Paul Erdős's Empty Hexagon Problem asks if there exists a number $H(6)$ such that for all sets of $n \geq H$ points in general position on the plane six of the points form the vertices of an empty convex hexagon. This problem is open.

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Acknowledgments

I would like to thank Grant for putting up with me. I would also like to thank my advisor and my second reader for bearing with me through this difficult semester. Of course, I would like to thank my parents for supporting me and making it possible for me to work.

Chapter 1

Preface

> *I looked at the book on my table that had stared at me like enemies a little while before. They were again the life of my life.* Ach! *Nothing was so beautiful as to learn, to know, to master by the sheer force of my will even the dead squares and triangles of geometry. I seized my books and hugged them to my breast as though they were living things.*
>
> Anzia Yezierska

The Erdős problem which concerns us in this paper first came to my attention one day when I was searching the Internet for a thesis problem. It had been contributed by Pavel Valtr to a set of problems for undergraduates. The problem's overt simplicity attracted me. I love to tell people about my work, so I wanted a problem that anyone could understand. I also knew when I chose the problem that like other plainly formulated open problems (Goldbach's conjecture comes to mind), the problem has a stubborn intractability which will seize the attention of anyone to whom I communicate it. Privately, as I have worked with it, I have learned to revel in its complex subtleties as well.

In addition to gaining knowledge about the problem and all of the areas of mathematics that it touches, I feel as if I have become familiar with the formulator of the problem to some extent. The original "Happy End" problem was formulated in 1935 by Esther Klein. Later, when the political situation in Hungary became hostile, the Szekeres couple moved to Australia where, I think, Erdős visited them in 1978. Seeing his friends again perhaps brought about the revisiting of the Happy End Problem and the creation of the Empty n-gons Problem.

The Empty n-gons Problem is the one I address in this paper. The question is asked, "For any natural number n does there exist a number $H(n)$

such that in any set of points in the plane such that no three are on a line with $H(n)$ or more points there will be an empty, convex n-gon?"

Although the requirement of emptiness was the next logical constraint on the problem, I am tempted to suggest its allegorical significance. Szekeres himself has remarked that Erdős may have also had an eye for Klein. It may be impertinent of me to draw this conclusion, but I believe that Erdős, in formulating this problem, was feeling his loneliness more sharply than usual. The emptiness of the convex n-gons seems to echo an emptiness in the heart.

When I originally began to work on the Erdős empty hexagons problem I fully intended to wield the battle ax of Gröbner bases at it, but after several weeks of trying to reformulate the geometric properties of a set without empty hexagons as polynomials, I came to my senses. The solution to this problem involves Gröbner bases about as much as the acquisition of a hummingbird involves a sledgehammer.

Thus, my progress has been slow. A lemma here, a conjecture there, and lots of observations, thanks to Mark Overmars's program "empty6." At this point, mid-way through the year, I am now approaching the problem as I would an onion: Literally peeling away the layers and often breaking out in tears.

Someone has asked why I chose a problem which has been so intractable for so many mathematicians. Frankly, I figured that everyone would agree that I could survey the work previously done on the problem and that they would expect not much more of me; I hope to surprise everyone. Morris and Soltan wrote in [19] that the Empty Hexagons Problem

> . . . remains a large gap that probably will require some new paradigms to be bridged.

Perhaps by the time this thesis is done I will have found their new "paradigms", but that is being optimistic–very optimistic.

Chapter 2

Problem Statement

Problems worthy of attack
Prove their worth by fighting back.

Paul Erdős

One day, in 1935, Esther Klein, a young Hungarian mathematics student, sat by a window in her parents' apartment doodling sets of points on a piece of paper. Eventually there formed in her mind a problem to this effect [19]:

Problem 2.1 Erdős-Szekeres Problem. *For any positive integer $n \geq 3$, determine the smallest positive integer $N(n)$ such that any set of at least $N(n)$ points in general position in the plane (that is, with no three points on a line) contains n points which are the vertices of a convex n-gon.*

On her own Klein found $N(3) = 3$ trivially. Then, by considering the cases where the convex hull of the set is a triangle, quadrilateral, or pentagon, proved to herself that no matter where the remaining points lie inside the convex hull, there would always be a convex quadrilateral (see Figure 2.1). Thus, the 4-gon conclusion is that $N(4) = 5$.

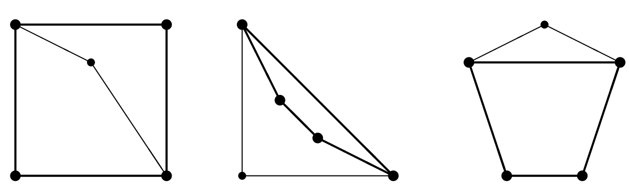

Figure 2.1: Pictorial proof that $N(4) = 5$.

The next day Klein was eager to share this new problem with her friends. George Szekeres, who would later admit that his enthusiasm for the problem was motivated to some extent by his interest in its creator [23], and Paul Erdős, on whom Szekeres projected the same motivation, went to work on it. Later that year Erdős and Szekeres published [12], which proved that the number $N(n)$ exists for every n and gave two proofs, one of which was founded on a theorem of Ramsey, the other being a combination of geometry and combinatorics. Not long after the publication of the paper Szekeres and Klein married, prompting Erős to dub the problem the "Happy End Problem."

It appears that Erdős visited the Szekereses and revisited the "Happy End Problem" in 1978. At the time, George and Esther were living in Sydney and the following problem appeared in the gazette of the Australian Mathematical Society [9].

Problem 2.2 Erdős Problem of Empty n-gons. *For any positive integer $n \geq 3$, determine the smallest positive integer $H(n)$, if it exists, such that any set x of at least $H(n)$ points in general position in the plane contains n points which are the vertices of a n-hole, that is, an* empty *convex polygon whose interior does not contain any point in X.*

The proof for the fact that $H(3) = 3$ is trivial and that $H(4) = 5$ can also be obtained from Figure 2.1. Later that year Harborth [15] proved that $H(5) = 10$, then five years passed until Horton proved that $H(n)$ does not exist for $n \geq 7$. These proofs and the techniques involved will be detailed in the next chapter. Although much work has been done by Valtr, Overmars, and many others to resolve the case $n = 6$, the question of even the existence of $H(6)$ remains open. It is my wish that the problem will be less open by the end of this thesis.

Chapter 3

Background Literature

So long as a man remains a gregarious and sociable being, he cannot cut himself off from the gratification of the instinct of imparting what he is learning . . .

J.J. Sylvester

Not long after Erdős posed the problem, mathematicians began to work on the Empty n-gons Problem. Erdős noted that Esther Klein's original proof for $N(4) = 5$ also suffices to prove that $H(4) = 5$. The proof is analogous to the proof of the fact that $N(4) = 5$ (see Figure 2.1).

3.1 The Erdős-Szekeres Theorem

Although the fact that a strict equality for $N(n)$ has not yet been found has not put a damper on the search for $H(n)$ for $n \neq 6$, it is useful to take note of the current bounds. The current tightest lower bound remains from Erdős and Szekeres's original 1939 paper, reprinted in [12], while the upper bound has been improved in a flurry of activity and seems to remain stubbornly where Tóth and Vlatr [24] left it. Thus, the current best bounds for the general case are

$$2^{n-2} + 1 \leq N(n) \leq \binom{2n-5}{n-3} + 2.$$

These bounds are not to be confused with the Erdős-Szekeres theorem which states that the number $N(n)$ must exist for all n. Szekeres has conjectured that the lower bound is actually an equality. As noted before, the equality

$N(4) = 5$ is known from Esther Klein's proof and it is noted by Erdős [12] that Endre Makai proved the equality $N(5) = 9$, by creating the only counterexample of eight points (see figure 3.1). With this arrangement of eight points it is impossible to place a ninth in general position without creating a convex pentagon.

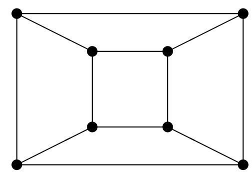

Figure 3.1: Makai's eight-point counterexample which leads to the conclusion that $N(5) = 9$.

3.2 Harborth and $H(5)$

In [15], Harborth proved that $H(5) = 10$. Comprehension of Harborth's proof relies heavily on the reader's abilities to visualize the point set at hand and to read German.[1] However, the proof can be summarized easily.

First, Figure 3.2 shows that if $H(5)$ exists, then it is certainly greater than 9. Then Harborth uses Makai's Erdős-Szekeres theorem-related result to say that since $N(5) = 9$ then for every set of points X_n for $n \geq 10$ there must be at least one convex pentagon which is not necessarily empty. If the pentagon is non-empty, then Harborth considers several cases which systematically narrow the margin in which $H(5)$ lies. First, if there are $m \geq 2$ points inside the pentagon, then we know that two of the m points and three of the points on the pentagon form another pentagon. If the smaller pentagon is non-empty, we find another two points inside that one, form another pentagon, and so on. Eventually we will find a convex pentagon P with either zero or one point inside it.

In the case where there is one point inside the convex pentagon, that one point forms quadrilaterals with the vertices of the pentagons. Only if all these quadrilaterals are convex is there an empty convex pentagon within the convex pentagon. Otherwise, Harborth dissects the plane into regions and shows that if there are points in any of those regions then there must be an empty convex pentagon in X_n. In the end, $H(5) = 10$.

[1]A literal, word-by-word translation can be found in my thesis notebook.

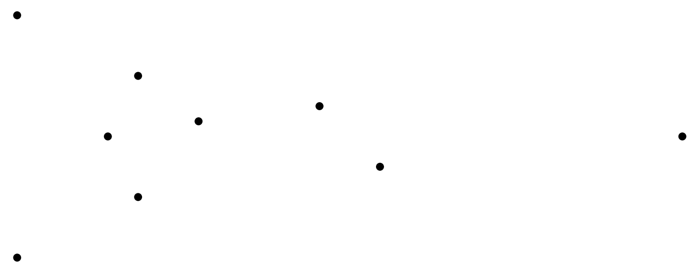

Figure 3.2: Heiko Harborth's set of nine points with no empty convex pentagons.

3.3 Horton's Construction for $n \geq 7$

In [17], J.D. Horton proved that for every $n \geq 7$, $H(n)$ does not exist by constructing arbitrarily large sets with no empty convex n-gons. This elegant work has proven to be very important to generalizations to higher dimensions.

In building his arbitrarily large sets, Horton created a set of points S_k such that $|S_k| = 2^k$. The x-coordinates of the points of the set are integers i such that $0 \leq i < 2^k$. The y-coordinates of the points are defined as $d(i) = \sum_{j=1}^{k} a_j c^{j-1}$ where $c = 2^k + 1$ and a_j is the jth digit in the binary representation of i such that $i = a_0 a_1 \ldots a_k$ *including* leading zeros.

Horton then goes on to observe several useful properties of the set S_k and uses those and the "caps and cups" method to show that any empty convex n-gon in the set has $n \leq 6$ points. Since the set S_k can be constructed to be as big as we like, $H(n)$ does not exist for $n \geq 7$.

3.4 Higher Dimensional Work

In attempting to get a bird's eye view on the Empty n-gons Problem, Vlatr, Harborth, and Bisztriczky have extended their thoughts to higher dimensions. As successful as these mathematicians were in their work, the existence and value of $H(6)$ still remains elusive.

3.4.1 Valtr's Generalization for $\mathbb{R}^d$

In [27] Valtr made a sweeping statement for higher dimensions. By extending both the Erdős-Szekeres theorem and Horton's construction to higher dimensions, Valtr shows that we can make bounds on the value of a number $n(d)$ such that $n(d)$ is the maximum number such that any sufficiently large set of points in $\mathbb{R}^d$ in strongly general position (no $k+1$ points determine a k-dimensional subspace for $k = 1, \ldots, d-1$) contains an empty convex polytope with $n(d)$ points for vertices. For $n \geq 2$ the bounds Valtr finds are

$$2d + 1 \leq n(d) \leq 2^{d-1} \left[\left(\prod_{i=1}^{d-1} p_i \right) + 1 \right]$$

where p_i is the ith prime.

Unfortunately, these bounds are not so kind as to give us an answer as to whether $H(6)$ exists or not: $5 \leq n(2) \leq 6$. Therefore, for a sufficiently large set of points, the largest empty convex n-gon we can be assured will exist in the set has either $n = 5$ or $n = 6$. We already know from Harborth's proof that we can guarantee that there will be an empty convex pentagon in the plane, but Valtr's bounds do *not* rule out the possibility that we might be able to promise the existence of an empty hexagon in sufficiently large sets of points in general position.

3.4.2 Harborth Returns

Four years after Valtr's success with creating bounds on $n(d)$, Harborth returned (in English, this time) to the scene with Bisztriczky [3]. Wielding Valtr's results, Gale's evenness condition (that a set of points V_d defines one of the facets of a cyclic d-dimensional polytope if and only if every two points of the polytope not in V_d are separated on the moment curve, $x(s) = (s, s^2, \ldots, s^d)$, by an even number of points on V_d), Radon's theorem (if V is a set of $d+2$ points in general position in d-dimensional space, then there exist unique disjoint subsets V' and V'' such that the intersection of their convex hulls is empty), and many other ideas, some of which were decidedly not of combinatorial nature, the two mathematicians announced that for a dimension d and integer n, the smallest integer $h(n, d)$, such that for any set of $h(n, d)$ points in general position in $\mathbb{R}^d$ there will be n points which form the vertices of an empty convex d-dimensional polytope, has the value

$$h(n, d) = d + 2k - 1$$

where $k = n - d$ for $1 \leq k \leq \lfloor d/2 \rfloor + 1$.

Unfortunately, this relation flails impotently at the empty hexagons problem since in the case for $n = 6$ and $d = 2$, $6 - 2 = 4 > 2$ and the necessary conditions on k are not satisfied. Therefore, we cannot even apply

Bisztriczky and Harborth's work to ascertain a bound on $H(6)$, if it does exist.

3.5 Algorithmic Methods and $H(6)$

Although it is a provable mathematical fact that $\infty - n = \infty$ for finite n, mathematicians have rejoiced at every new largest set of n points without empty convex hexagons.

Since it can be very difficult for human beings to look at a set of points and tell whether or not there is a subset which is an empty convex hexagon, several computer scientists have taken up the challenge of creating algorithms which will detect empty hexagons.

Avis and Rappaport [1] were among the first to ply their art on the problem and implemented an algorithm which found a set of points in general position with $|X_n| = 20$ and no empty convex hexagons. Their algorithm used some preprocessing but was primarily based on a modification of an algorithm of Chvátal and Klincsek [7] which involves defining a convex polygon in terms of a "fan" within a visibility graph. The question the Avis and Rappaport algorithm answers is, "What is the largest empty convex subset of the the set X_n?" The runtime complexity of Avis and Rappaport's algorithm is $O(n^2)$ where n is the number of points in the set in question.

Another computer scientist who has employed the idea of a visibility graph is Mark Overmars. In [21] Overmars, Scholten, and Vincent created an algorithm which answers the question, "Given a set V of points in general position with no empty convex hexagons and a point u, does the set $V \cup u$ contain empty convex hexagons?" The algorithm is $O(n^2)$ where n is the number of points in the set in question, but it should be noted that their algorithm in practice runs faster than the algorithm of Avis and Rappaport. The largest set of points generated by this algorithm when random points were chosen was 18, but when an incremental "backtracking" approach was used, this was improved to 26 points. Unfortunately, [21] was never accepted for publication, but Overmars has had the foresight to supply the interested reader with a copy on his website.

The proof of the algorithm in [21] contains some useful Harborth-type observations concerning the set V with no empty convex hexagons and a point u such that $V \cup u$ has empty convex hexagons. These observations will be introduced as they become useful in the proofs section.

In [20] Overmars revisited the algorithm from [21] and made some improvements which included using Brownian motion and backtracking as heuristic methods for choosing the additional point u. With these improvements and a state of the art computer, Overmars found an empty convex hexagon-free set with 29 points, implying that if $H(6)$ exists, then $H(6) \geq 30$.

Overmars's 29-point set is currently the largest found yet and may be seen in Figure 3.3.

Figure 3.3: Mark Overmars's set of 29 points with no empty hexagons.

Besides announcing the largest 6-hole-free set in [20], Overmars announced some intriguing conjectures. Let the set points C_n on the convex hull of a set X be the points that a rubber band would touch if we were to stretch it around the whole set tightly. Overmars conjectured that any set of points on the plane in general position, X, with no empty convex hexagons would have $|C_n| < 8$. Although this would have been an exciting and useful conjecture to prove, I wasted the three weeks I spent trying to prove it. In this thesis I provide a counterexample to this conjecture. It is a set of points with no empty convex hexagons and has a convex hull of eight points (see Figure 3.4).

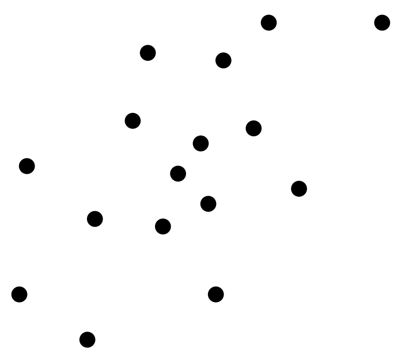

Figure 3.4: A set of points in general position with no empty hexagons and a convex hull of eight points.

If Overmars's conjecture had been true, it would have had several interesting implications. First, let a layer of a set X_n be the set of points which lie on the convex hull of the set $X_n \setminus C_n$. In other words, if we had our rubber band stretched around the set X_n and then simultaneously removed all the points it touched, the new points around which the rubber band would tighten is the second layer L_2 ($C_n = L_1$, as with an onion).

Now, if we consider Overmars's conjecture more closely, we will see that if it were true, then the number of layers in any set of points in general position with no empty convex hexagons would be linear in the number of points. It is also trivial to prove that the number of layers in such a set would be bounded below by $\log n$ where n is the number of points in the set. Overmars's conjecture also has some other interesting implications, but I will elucidate those when an altered conjecture is introduced in the next chapter.

3.5.1 Valtr's Construction

In the same year that Harborth and Bisztriczky published their results, Valtr improved the upper bounds of some results of Bárány and Füredi [2] which had to do with the minimum numbers of empty polygons in planar sets.

To give the upper bounds he reports, Valtr constructs a set of points related to Horton's S_k. Among other bounds for the numbers of n-gons where $n = 3, 4, 5$, this n-point set has fewer than $\frac{n^2}{3}$. Of course, no lower bound is given because such a lower bound would involve $H(6)$, which we do not know.

3.6 Summary

The easiest way to summarize the work done to ascertain $H(n)$ is to say "valiant." However, in spite of all the efforts to chip away at the problem, $H(6)$ remains as elusive as ever.

Chapter 4

Protoproof

A mathematician is a machine for turning coffee into theorems.

Alfréd Rényi

Weak coffee is only fit for lemmas.

Paul Turàn

Over the past few months I have been toying with a revision of Overmar's conjecture that there can be no more than 7 points on the boundary of the convex hull of a set of points in general position with no empty hexagons. In short, the revised conjecture insists that there can be no more than 8 points that define the convex hull of a general-position point set with no 6-holes. In the first section of this chapter I will give a general outline of the proposed proof to motivate the introduction of some very, very new concepts. Later in the chapter, after the essential concepts have been thoroughly explained, detailed proofs will be presented of each of the essential lemmas.

4.1 Proposed Proof Outline

Conjecture 4.1 *(Existence and Bound on $H(6)$) There exists a number $H(6)$ such that a set of points in general position with $H(6)$ or more points will certainly contain a $6-hole$. Additionally, $H(6) \leq 1718$.*

Proof. The proof of this conjecture follows immediately from Valtr and Tóth's bounds [24] on $N(9)$ if the following nested lemmas are true.

Before the first lemma is introduced, it must be prefaced with the definition of a concept which will allow us to think about where were cannot place additional points when we are aiming to avoid creating 6-holes.

A word on notation: From this point on, vertices of polygons will be labelled counterclockwise. Also, a half-plane will be defined by an ordered pair (p_i, p_j) and will consist of the open half-plane that would lie to one's left if one were walking from point p_i to p_j along the line segment connecting them. In this way, $(p_i, p_j) \cup (p_j, p_i)$ is the whole plane without the line going through p_i and p_j.

Definition 4.2 *Let $P = \{p_1, p_2, p_3, p_4, p_5\}$ be a 5-hole. The* forbidden areas *of P is*

$$F = \bigcup_{i \in [5]} ((p_{i-1}, p_i) \cap (p_{i+1}, p_{i+2}) \cap (p_{i+1}, p_i)). \tag{4.1.1}$$

To get a more visual idea of where the forbidden areas of a 5-hole are, refer to Figure 4.1.

The idea of forbidden areas is important because it allows us to check ourselves from setting an additional point which may cause a 6-hole in a pointset that previously had none. In particular, we use the idea in our first lemma to conclude that we cannot add a point to a set with a convex hull defined by eight points such that the enclosing 8-gon becomes a 9-gon.

Lemma 4.3 *(Unproven) For every set of points in general position with a convex 8-gon, the forbidden areas of the 5-holes within the convex 8-gon will cover the areas in which one could place a point to make the 8-gon into a 9-gon.*

This lemma was inspired by Figures 4.2 and 4.3 where the forbidden areas (in black) of the 5-holes in the pointsets cover the areas in which points could be added so that the convex hull is defined by nine points rather than eight. These figures were generated using Mark Overmars' *empty6* program.

This lemma depends heavily on the lemma which follows. Given the next lemma, however, the proof is still not trivial. It will still require a new concept which will be introduced in Section 4.2.2.

The next lemma is the wobbly table upon which this house of cards is built. However, as with the previous lemma, its introduction must be preceded by the definition of a *layer* and what it means for a point set to be *unique up to taking layers*.

Definition 4.4 *Let X be a set of points on the plane. The first* layer *of X is the set of points $L_1 \subseteq X$ such that $L_1 \subseteq \partial conv(X)$.*

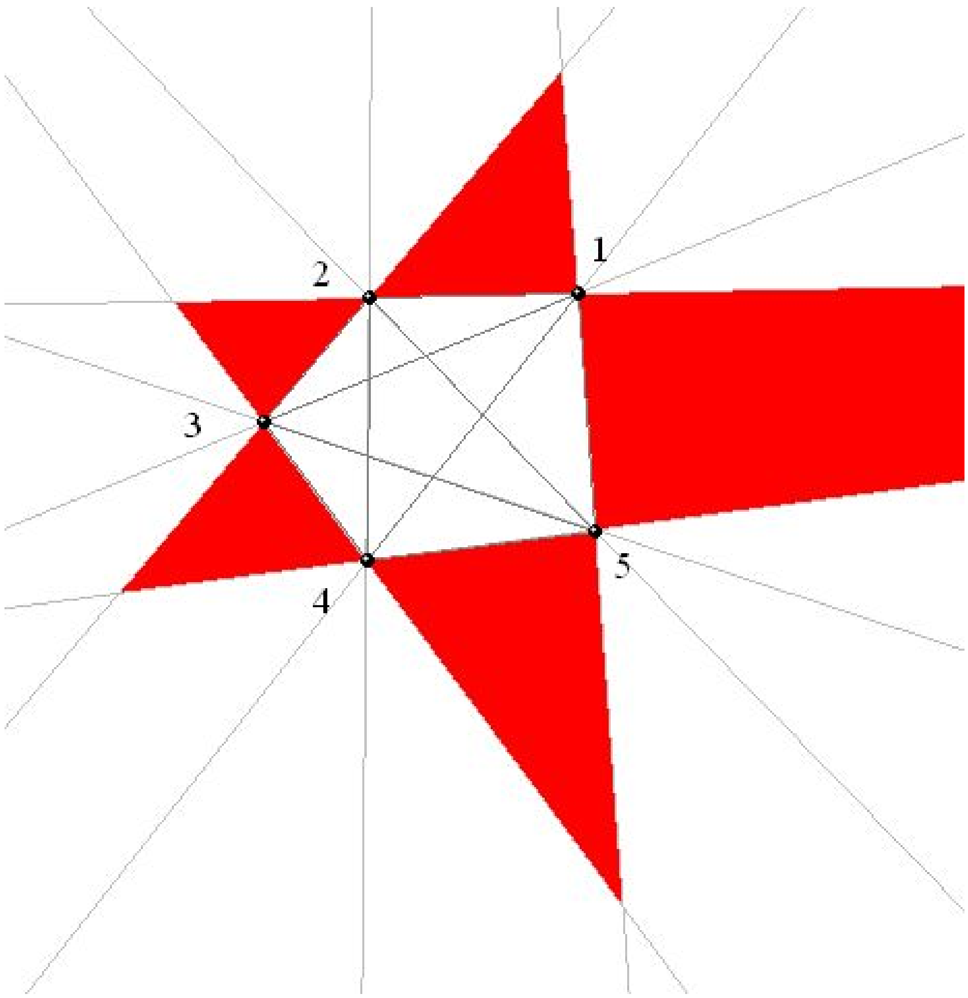

Figure 4.1: A 5-hole and its forbidden areas.

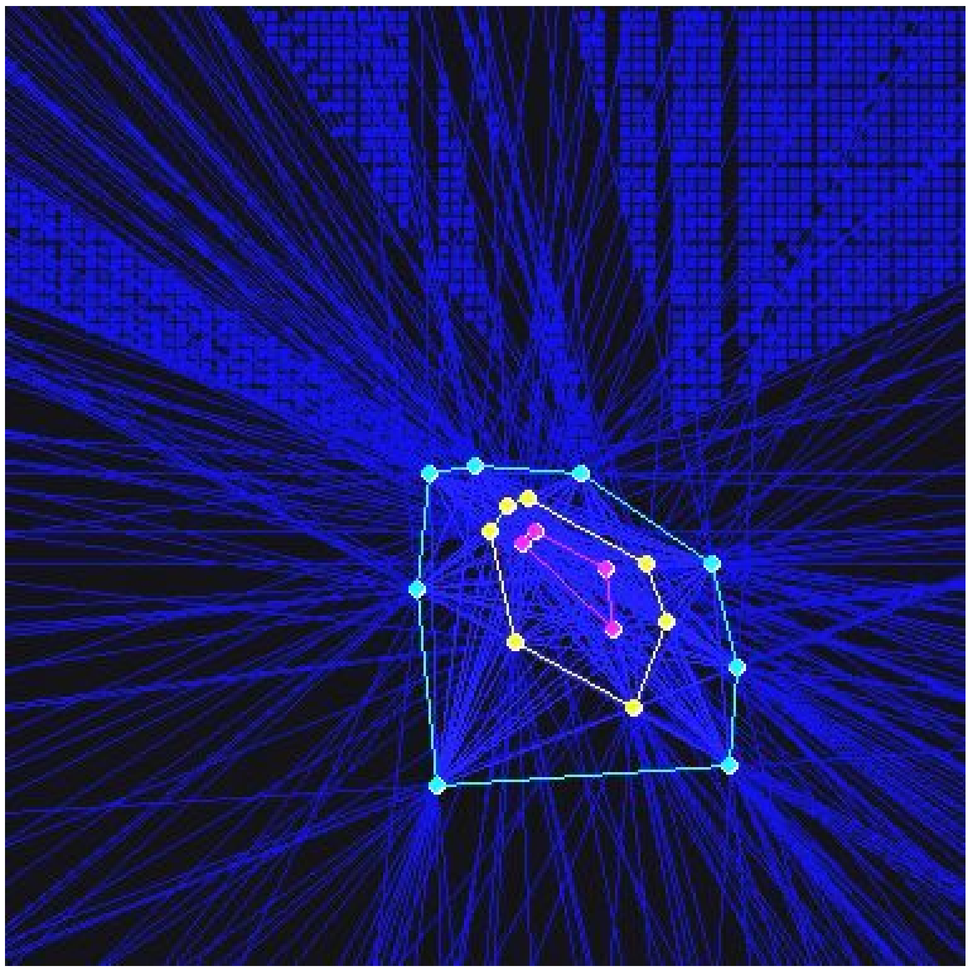

Figure 4.2: A set of points with 8 points defining the convex hull.

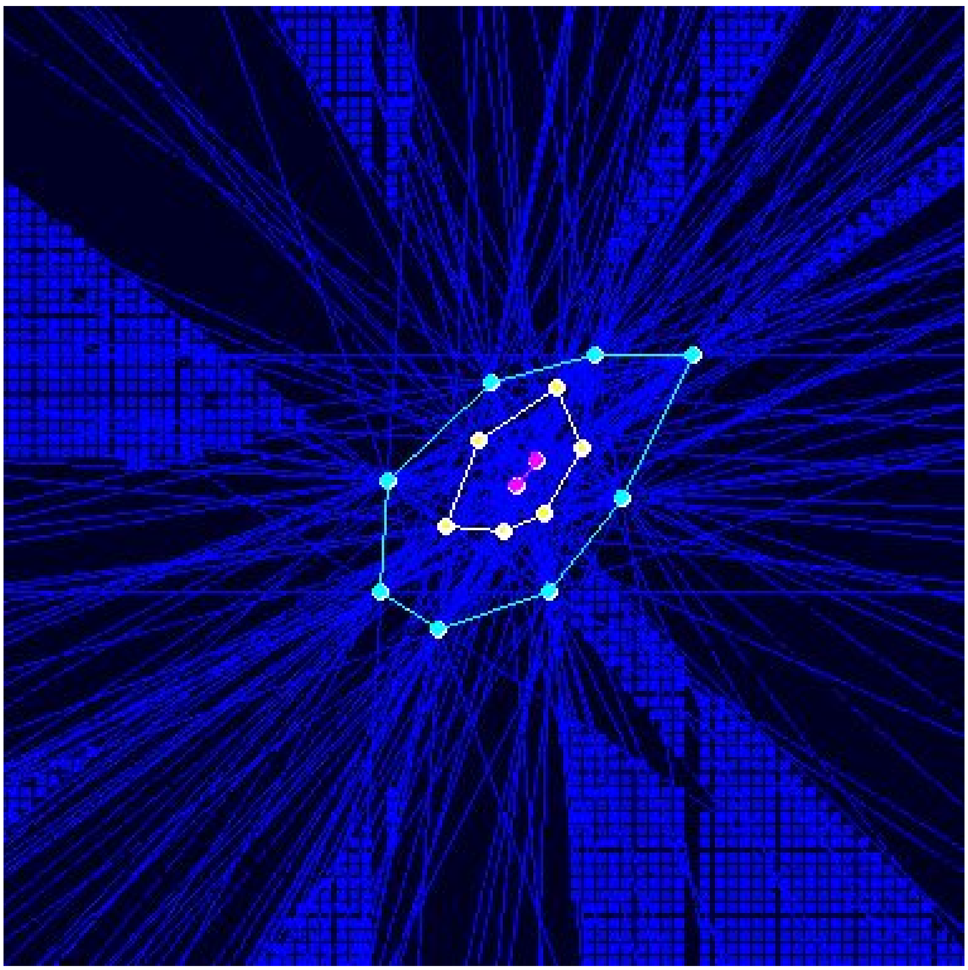

Figure 4.3: Another set of points with 8 points defining the convex hull.

That is, the first convex layer of a pointset X are the points in X which lie on the boundary of the convex hull of X: A rubber band tightened around X will touch the first convex layer and nothing else. Subsequent convex layers are defined recursively.

Definition 4.5 *Let X be a set of points on the plane. The k*th convex layer *of X is the set of points $L_k \subseteq X$ such that $L_k \subseteq \partial conv(X\ (\cup_{i=1}^{k-1} L_k))$.*

Figures 4.2 and 4.3 illustrate layers. The cyan, yellow, and magenta connected points are the first, second, and third layers respectively.

The definition of layers now allows us to create equivalency classes among sets of points with a fixed number of points defining their convex hulls.

Definition 4.6 *Two point sets, X and Y whose first layers possess the same number of points are of the same layer equivalency class if*

1. *the number of layers, ℓ, in X is equal to the number of layers in Y, and*

2. *$|L_{k,X}| = |L_{k,Y}|$ for all $k \in [\ell]$.*

We can consider a representative of such a layer equivalency class as *unique up to taking layers*. For example, the pointsets in Figures 4.2 and 4.3 are representatives of the $\{8, 7, 4\}$ and $\{8, 6, 2\}$ layer equivalency classes, respectively. This conjecture arose after I logged many hours just watching Overmars' program generate sets of points in general position with no empty hexagons, and these are the only two equivalency classes which have appeared with 8 points on a layer. This concept allows us to develop the following lemma.

Lemma 4.7 *(Unproven) There are a finitely many arrangements of points in general position with eight points in the first layer and no 6-holes. In fact, there are two which are unique up to taking layers.*

Since the proof of this lemma also requires some very new, very complicated ideas, I will put it off until those ideas have been fully explained. Specifically, this lemma is proven using ideas in Section 4.2.1.

4.2 New Concepts

Although I have loved geometry dearly since I received my first stencil set of geometric figures for my sixth birthday, I have found that thinking about things in terms of graphs comes most naturally to me. Thus, I have chosen to translate the problems at hand into problems of Graph Theory and Ramsey Theory. I do not know if the techniques set forth here are original, but they are new to me. If it does turn out that they are original, I can only hope that for the trouble they have been to write down, they will be of use to others.

4.2.1 Half-Plane Intersection Graphs

Wishing to prove Lemma 4.7, I have attempted to devise a device for transplanting the problem into graph theory.

When we are given a set of points X, we can labels the points $[|X|]$ and, given the definition of a half-plane (p_i, p_j) as the half-plane to one's left as one walks from point p_i to point p_j, we can then put the half-planes in lexicographical order:

$$(p_{i_1}, p_{j_1}) \prec (p_{i_2}, p_{j_2}) \text{ iff } \begin{cases} i_1 < i_2 \\ i_1 = i_2,\ j_1 < j_2 \end{cases} \tag{4.2.1}$$

Using this ordering, we can easily enumerate the half-planes. For a set of $|X|$ points, the half-plane (p_i, p_j) will be labelled

$$H_{(i-1)|X|+j}. \tag{4.2.2}$$

The astute reader might note that this enumeration assigns labels to a few apparently superfluous half-planes. For example, the plane (p_1, p_1) is counted as H_1. Although this particular degenerate half-plane *is* completely useless, it makes programming significantly less complicated if we simply abide its presence. In the future it may also turn out that these degenerate half-planes are of some use–although the use deftly escapes me at this time.

Definition 4.8 *Given a point-set X, let I be the* half-plane intersection matrix *such that if the half-planes $H_a = (p_{i_a}, p_{j_a})$ and $H_b = (p_{i_b}, p_{j_b})$ intersect to form a convex area, then $I_{a,b} = 1$ and otherwise $I_{a,b} = 0$*

At this point the reader may be asking just what I mean by two half-planes enclosing a convex area–surely the intersection of two convex bodies is also convex?

Recall, however, that my proof requires a severely restricted world-view. First of all, we are concerned only with intersections inside our n-gon. Second, we require that if the two segments defining H_a and H_b cross, then their intersection is not considered convex. Table **??** is a summary of cases of

4.2.2 Forbidden Graphs

Chapter 5

Conclusions

A conclusion is the place where you got tired of thinking.

Anonymous

My latest conjecture, that the upper bound of $N(9)$ serves also as an upper bound for $H(6)$, is extremely promising. A proof for this conjecture will probably vaguely resemble Harborth's $H(5)$ proof. If I can succeed in proving even just that conjecture, I will have made a fairly large contribution to the work on the problem.

I am very happy with the progress I am making. I wish I could dedicate all of my time to my thesis.

Appendix A

Appendices

A mathematician's work is mostly a tangle of guesswork, analogy, wishful thinking, and frustration, and proof, far from being the core of discovery, is wore often than not a way of making sure that our minds are not playing tricks.

–Gian-Carlo Rota

Insert notes and trivial things which might ruin the flow of the paper.

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