

Research Proposal: Alexander Polynomials of Tunnel Number One Knots

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1. Prior Work

Tunnel number one knots are defined by Hilden et al [7] as follows:

Think of a classical knot as a solid torus embedded in S^3 . If we add a thickened arc to the torus, we have an embedded genus two handlebody. If this can be done in such a way that the complement is also a genus two handlebody then the original knot is called a tunnel number one knot.

Two-bridge knots are a totally understood and classified set of alternating knots which are a subset of tunnel number one knots. For every tunnel number one knot K , the group $\Pi_1(S^3 - K) = \langle a, b \mid aw = wb \rangle$ where w is a word in a and b depending on K . For the two-bridge knots $K_{(p,q)}$ there is a simple formula for w . For tunnel number one knots, there is no known formula for w , but w has a similar form.

It is conjectured that for all alternating knots K , with Alexander polynomial $\Delta_K(t)$, the polynomial $\Delta_K(-t)$ is “trapezoidal” (the coefficients increase, then remain constant, then decrease). It has been proved that the conjecture is true for arborescent knots, a larger class than two-bridge knots.

The number of distinct two-bridge knots of a given crossing number has been counted by Ernst and Sumners [5], and this provided the first proof that the number of prime knots increases exponentially with crossing number. It is almost certain that the ratio of the number of distinct Alexander polynomials to the number of distinct knots approaches zero as the crossing number goes to infinity, but this remains unknown.

2. Proposed Research

There are many interesting open problems involving tunnel number one knots. The theorem above about the group presentation for tunnel number one knots was only recently proved (2001), and it may be useful in proving other results about tunnel number one knots or shedding more light on two-bridge knots. I will see what new information I can discover about tunnel number one knots using this group presentation. In particular, I will try to prove the conjecture about trapezoidal polynomials $\Delta_K(-t)$ for alternating tunnel number one knots. The Alexander polynomial can be obtained from the group presentation by using Fox's calculus. This approach is particularly simple for 2-generator, 1-relator groups. I will also use the group presentation approach to look for simpler proofs to some of what is already known about Alexander polynomials of two-bridge knots. In particular, I will attempt to count the Alexander polynomials for two-bridge knots of a given crossing number, and verify for two-bridge knots that the number

of distinct Alexander polynomials to the number of distinct knots approaches zero as the crossing number goes to infinity.

3. Intended Reading

I have no background in knot theory, and have never taken a knot theory class. I intend to do some background reading over the summer. The readings include the following:

Books:

- [1] Adams, Colin. *The Knot Book*. 1994
- [2] Crowell, Richard; Fox, Ralph. *Introduction to Knot Theory*. 1977
- [3] Livingston, Charles. *Knot Theory*. 1993
- [4] Rolfsen, Dale. *Knots and Links*. 1990

Articles:

- [5] Ernst, C.; Sumners, D. W. "The growth of the number of prime knots. *Math. Proc. Cambridge Philos. Soc.* **102** (1987), no. 2, 303--315
- [6] Hartley, Richard. "On 2-Bridge Knot Polynomials." *J. Austral. Math. Soc.* 28 (1979) 241-249
- [7] Hilden, Hugh; Tejada, Debora; Toro, Margarita. "Tunnel Number One Knots Have Palindrome Presentations." [*J. Knot Theory Ramifications* **11** \(2002\), no. 5, 815--831.](#)
- [8] Murasugi, Kunio. "On the Alexander Polynomial of Alternating Algebraic Knots." [*J. Austral. Math. Soc. Ser. A* **39** \(1985\), no. 3, 317--333.](#)