Harvey Mudd College
Mathematics

Senior Thesis
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Research Proposal: Pebbling Results and Graph Analogues of the KKM Lemma

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1 Introduction

In the summer of 2004, I participated in an REU program at East Tennessee State University. Many of the investigations I worked on were successful, leading to 4-6 papers, some of which have been submitted, and others are still in the writing process. There are many unanswered questions left in the investigations that I was conducting, and some of the work already done is not complete. Given that my original thesis topic also is in graph theory, my new intention is to work simultaneously on questions based upon my REU research as well as with the previous problem I desired to tackle. This proposal will describe each of the these general areas.

2 Proposed Research: The KKM Lemma on Graphs

In 1929, Knaster, Kuratowski and Mazurkiewicz proved the following lemma, [4] which has admitted many combinatorial generalizations. Let $C_1, \ldots, C_{d+1}$ be closed sets that cover a d+1-simplex, which is the generalization of a triangle in $d+1$ dimensions, such that each face, $< i_1, \ldots, i_k >$ is covered by the sets $C_1, \ldots, C_k$. Call this a KKM cover. Then the intersection of all the C_i is nonempty. Recent work has been done by Niedermaier and Su [5] on applying this theorem to graphs, particularly labelled trees and cycles.

I intend to generalize the following results of [5] for other families of graphs: Let T be a tree whose leaves are labelled, $1, 2, \ldots, n$. Let C_i be a path starting at the ith leaf. Condition the C_i such that for any two leaves, i and j, there exists a path from i to j along $C_i \cup C_j$. Given this criterion, there exists some point p in the tree where all the paths intersect. We can describe a similar condition for cycle graphs. Let G be a cycle graph with k vertices. For each vertex, construct a path H_i, so that there exists a path between any two arbitrary vertices i and j using only H_i and H_j. In this case, there exists a point where at least $\lfloor \frac{k}{2} \rfloor + 1$ vertices intersect. A graph is k-edge-connected if the removal of $k-1$ vertices leaves the graph still connected. Notice that a tree is 1-edge-connected, and a cycle is 2-edge-connected. I hope to generalize the preceding results to certain families of 3-edge-connected graphs, such as the 3-cube, and conjecture that in the 3-cube, there exists a point where at least $\lfloor \frac{k}{3} \rfloor + 1$ vertices intersect given an appropriate creation of sets. It may be possible to generalize the result even further to the n-cube since a tree is actually a generalization of a 1-cube, and a cycle is simply a generalization of a 2-cube.

There are other types of grahps that have yet to be examined that could produce new results. One such family of 3-edge-connected graphs that interests me are the wheel graphs, which consist of a cycle with a vertex inside the cycle that connects to all the outside vertices of the cycle. Perhaps this is the correct generalization instead of the n-cube. Directed graphs have also not be addressed, and there may be some analogous theorems. It may also be possible to consider applications of the lemma where the vertices correspond to elements of finite groups instead of integers. This aspect of the problem has not been considered before and may provide additional insight into the deeper workings of the KKM lemma.

3 Proposed Research: Graph Pebbling

One recent development in graph theory, suggested by Lagarias and Saks, called pebbling, has been the subject of much research and substantive generalizations. It was first introduced into the literature by Chung[1], and has been developed by many others including Hurlbert, who published a survey of pebbling results in [3]. Given a connected graph G, distribute k pebbles on its vertices in some configuration, C. A *pebbling move* is defined as the simultaneous removal of two pebbles from some vertex and addition of one pebble on an adjacent vertex. A pebble can be moved to a root vertex v if it is possible to place one pebble on v in a sequence of pebbling moves. We define the pebbling number, $\pi(G)$ to be the minimum number of pebbles needed so that for any initial distribution of pebbles, it is possible to move to any root vertex v in G.

The concept of cover solvability was introduced in [2]. We call a configuration on a graph *cover solvable* if, starting with this configuration, it is possible, through a sequence of pebbling moves, to simultaneously place one pebble on every vertex of the graph. The *cover pebbling number* of a graph, $\gamma(G)$, is defined as the smallest number such that every configuration of this size is cover solvable. At the REU, some results regarding cover pebbling and extensions of cover pebbling have been produced. My goal for this thesis is to extend some of these results, finish results that had complications, tighten up first drafts of proofs, and examine other generalizations of pebbling.

4 Prior Research

I have taken or placed out of the following classes that will by helpful in my research: Graph Theory, Discrete Mathematics, Geometric Combinatorics, Abstract Algebra I and II, and Independent Study in Graph Theory. I have also taught various topics in graph theory and other areas of discrete mathematics at the Hampshire College Summer Studies in Mathematics program during the summer of 2002. My research experience at both the University of Idaho and East Tennessee State University REU's will also be helpful during the thesis writing process.

References

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[4] B. Knaster, C. Kuratowski, and S. Mazurkiewicz, *Ein Beweis des Fixpunktsatzes fur n-dimensionale Simplexe*, Fund. Math. 14 (1929), 132137.

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[6] Z. F. Yang, *Multipermutation-Based Intersection Theorem and its Applications*. Journal of Optimization Theory and Applications. 104 (2000), 477-487.