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Research Proposal: Decimation-in-Frequency Fast Fourier Transforms for the Symmetric Group

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1 Introduction

In recent years, researchers have begun to approach Fourier transforms from a more algebraic perspective. For example, the classical Discrete Fourier Transform (DFT) takes n evenly spaced samples of a periodic function and transforms them into n amplitudes for different frequencies, called the Fourier coefficients of the sample data. If the data sample is considered as a function on the group $\mathbb{Z}/n\mathbb{Z}$, that is, as an element of the group ring $\mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$, then the DFT becomes an isomorphism of $\mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$ into the algebra of complex diagonal $n \times n$ matrices. The elements along the diagonal of the matrix corresponding to a given $f \in \mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$ represent the Fourier coefficients of f [3].

Wedderburn's Theorem states that, for any finite group G, $\mathbb{C}G$ is isomorphic to some block diagonal matrix algebra $\bigoplus_{j=1}^{h} \mathbb{C}^{d_j \times d_j}$. It is therefore possible to construct a generalization of the usual DFT for any finite group. DFTs for infinite groups (such as Lie groups) can also be defined, although there must be a way to sample the function on the group.

A Fast Fourier Transform (FFT) is any efficient implementation of a DFT. The classic example of an FFT is the Cooley-Tukey FFT for the traditional DFT, which computes the DFT in $O(N \log N)$ time, rather than the $O(N^2)$ time required by a naive implementation of the DFT. Similar techniques have yielded FFTs for all abelian groups G of complexity $O(|G| \log |G|)$ [3]. Determining the upper bound for the complexity of FFTs on an arbitrary finite group G remains an open problem, although a number of conjectures regarding this bound exist.

Most existing generalized FFTs follow the Cooley-Tukey algorithm in using a *decimation-in-time* decomposition. This approach constructs a series of coset partitions of the group from a chain of subgroups and then computes smaller transforms for each level of the partition. A different approach, effectively dual to the decimation-in-time method, is the *decimation-in-frequency* FFT, in which the frequency space is decomposed instead of the group. This decomposition may be accomplished by successively projecting the data into different subspaces to construct increasingly finer frequency information. As decimation-in-time algorithms have become increasingly difficult to improve, more attention has focused on the alternatives offered by decimation-in-frequency approaches [14].

2 Proposed Research

I propose to study the theory of decimation-in-frequency FFTs for the symmetric group, S_n, and to develop implementations of these FFTs. While some initial decimation-in-frequency algorithms for symmetric group FFTs have been developed, there is still a large amount of work to be done in the field. In particular, the Gentleman-Sande FFT on $\mathbb{Z}/n\mathbb{Z}$ remains the only decimation-in-frequency algorithm that is well understood algebraically. The symmetric group represents an ideal candidate for the further development of this algebraic theory to the noncommutative domain, as its matrix representation theory has been studied extensively in the literature [7]. Ideally, it will be possible to identify and to exploit the structure of the matrix representations and of the subgroups of S_n to construct decimation-in-frequency FFTs with efficiency matching or exceeding those of the current $O(n^2|S_n|)$ decimation-in-time FFTs [8].

In addition, any theoretical advances in the understanding of FFTs for the symmetric group may also apply to other groups of similar structure. One key feature that affords a convenient decomposition of the symmetric group is its multiplicity-free character graph. If we can understand the role this plays in FFTs on S_n, we may be able to construct similar FFTs for other groups that exhibit this character graph structure, such as Weyl groups or Coxeter groups [11, 13]. I therefore plan to investigate such generalizations of any theoretical results I obtain for the symmetric group.

3 Background Work

My background for this project derives from several classes. Introduction to Systems Engineering (Engr 59) first introduced me to the Fourier transform, the DFT, and the FFT. Abstract Algebra I (Math 171) has provided extensive background in group and ring theory, and Abstract Algebra II (Math 172) this semester builds on that knowledge with its emphasis on module theory and representation theory of groups. Furthermore, Abstract II has provided an introduction to the theory underlying the generalized FFTs that I intend to investigate in greater depth. In addition, Numerical Analysis (Math 165) exposes me to issues of computation accuracy and efficiency that will play a key role in the implementation of these algorithms. Finally, Applied Analysis (Math 180) and Complex Variables (Math 136) have both increased my familiarity with the Fourier transform, and several physics courses (Phys 116 and 161) have given me an appreciation of its physical applications and interpretations.

4 Current and Intended Reading

For Math 171 and 172, I have read extensively in Dummit and Foote [5], including sections on group theory, module theory, and group representation theory. Another excellent reference on representation theory is Serre's classic text. [15]. Gross [6] provides an overview of the historical development of the field of noncommutative harmonic analysis, including FFTs on groups. Clausen and Baum's *Fast Fourier Transforms* [3] provides a more detailed introduction to group FFTs, and papers by Rockmore, Orrison, and Maslen [9, 10, 12, 14] give further information on both decimation-in-time and decimation-in-frequency techniques.

There exists a wealth of information on the representation theory of the symmetric group and on its decimation-in-time FFTs. James and Kerber [7] is a well-known reference on both the character theory and matrix representation theory of S_n. Maslen, Clausen, and Baum have published a series of competing papers on decimation-in-time algorithms for such groups [1, 4, 8], the most recent ofo which present a bound of $O(n^2|S_n|)$ on the linear complexity of such algorithms.

Finally, other papers seek to generalize the methods used to analyze the representations of the symmetric group to other classes of groups, including Weyl groups and Iwahori-Hecke algebras. These include papers by Clausen [2], by Okounkov and Vershik [11], and by Ram [13]. They will be useful in any investigations of such groups and algebras and also as a way to gain further perspective and insight into the decomposition of the symmetric group.

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