

# Research Proposal: Combinatorial Consequences of LSB-Related Results

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## 1 Introduction

Often theorems which are powerful in their own specific fields have extensions or equivalences in other fields of mathematics. Connections between theorems in a particular field can often lend insight to the discovery or exploration of relationships between their ostensibly dissimilar incarnations in other fields.

The Kneser Conjecture is a powerful result in combinatorics. Kneser conjectured that in order to partition the  $n$ -element subsets of a  $(2n + k)$ -element set into classes such that any two subsets of the same class have a non-empty intersection, at least  $k + 2$  classes are necessary.

Greene recently used a slight generalization of the LSB-Theorem (a sort of topological fixed-point theorem) to substantially improve on Bárány's classic and widely heralded proof of Kneser's conjecture.[2]

## 2 Proposed Research

This thesis project will investigate the combinatorial consequences of relatives of the LSB-theorem. The LSB-theorem states that for any covering of the unit sphere in  $\mathbb{R}^{m+1}$  with  $m + 1$  or fewer closed sets, one of the sets must contain a pair of antipodes.[2]

Two close relatives of LSB within topology are Fan's generalization of LSB (where the conditions are more general and LSB emerges as a special case) and the KKM Lemma. The latter deals with special coverings of the simplex and can be shown to directly follow from LSB [1].

One of the main strategies for my research will be investigating the relationships between these related theorems in order to try to extend the approach that Greene and others use in establishing combinatorial-type results.

I propose to investigate the following questions:

- How can Fan's generalization of the LSB theorem be used combinatorially?
- Do existing proofs of implications within and between these theorems provide insight towards alternate proofs of Kneser's Conjecture?
- Given the tie between the LSB Theorem and KKM Lemma, can the KKM Lemma be used to elegantly justify a similarly important combinatorial result? How might such a result relate to Kneser's Conjecture?
- What generalizations of other related theorems in different areas of mathematics (Brouwer fixed point theorem, Tucker's Lemma, Sperner's Lemma, etc) might be useful in pursuing combinatorial results? How might constructive proofs of Sperner-type results play out in combinatorial settings?

### 3 Prior Research

This past summer I conducted research with Professor Francis Su on the relationship between the KKM Lemma and the LSB Theorem. This project produced a direct proof of KKM that uses the LSB Theorem [1]. This research expanded on some of the material covered in a Geometric Combinatorics class I took last Spring. I also had the opportunity to attend the Park City Math Institute conference on the same topic. I have some exposure to combinatorial proof techniques from a number theory course and a discrete mathematics course, as well as from recreational interest in these types of topics.

### References

- [1] Spencer, Gwen, and Su, Francis Edward Su, *The LSB Theorem Implies The KKM Lemma*, arXiv.org.
- [2] Greene, Joshua E., *A New Short Proof of Kneser's Conjecture*, American Mathematical Monthly, 109 (2002) pp. 918-920.

- [3] Su, Francis Edward, *Borsuk-Ulam Implies Brouwer: A Direct Construction*, American Mathematical Monthly, 104 (1997), pp. 855-859.