Solving for Volume-Minimizing Cycles in G_2-Manifolds

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May 3, 2005

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Abstract

M-theory, a generalization of string theory, motivates the search for examples of volume-minimizing cycles in Riemannian manifolds of G_2-holonomy. Methods of calibrated geometry lead to a system of four coupled nonlinear partial differential equations whose solutions correspond to associative submanifolds of $\mathbb{R}^7$, which are 3-dimensional and minimize volume in their real homology classes. Several approaches to finding new solutions are investigated, the most interesting of which exploits the quaternionic structure of the PDE system. A number of examples of associative 3-planes are explicitly given; these may possibly be projected to nontrivial volume-minimizing cycles in, for example, the G_2-manifold $\mathbb{R}^6 \times S^1$.

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Acknowledgments

First and foremost, I'd like to express deep gratitude to my advisor, Professor Weiqing Gu. This endeavor would not be what it is without her constant time, guidance, and thoughtful feedback. Additionally, I'd like to thank my second reader, Professor Jon Jacobsen, for his invaluable feedback, and the HMC Math senior thesis coordinator, Professor L.G. de Pillis.

Chapter 1

Introduction

1.1 Physics Background

Twentieth century physics produced two rather powerful theories of the natural world: general relativity and quantum mechanics. The first is a theory of gravity and space-time and typically only describes large-scale phenomena; the second tends to manifest only on a tiny length scale. Quantum mechanics was unified with special relativity to produce quantum field theories. Such theories have been successful at modeling three of the four fundamental forces of nature: electromagnetism and the strong and weak forces. However, attempts to unite *general* relativity with the other three forces into a single theory have yet to succeed. At this point in time, our best understanding of theoretical physics presents itself in two mutually exclusive frameworks: one for gravity, and one for everything else.

The idea of "string theory" has been around since the 1970s. Its premise is that fundamental particles are not the zero-dimensional points described by quantum field theories, but are actually one-dimensional pieces of vibrating strings. String theory was not recognized as potentially useful until the idea of supersymmetry was incorporated, producing "superstring theory." Several distinct versions of these theories slowly cropped up, and each seemed to offer the potential to describe all four fundamental forces, which would be a unification of the laws of nature.

However, the fact that there were different variations of the theory was a huge problem. To be successful, a so-called "theory of everything" would be unique. It would show why nature is precisely the way it is, and why the world could not be any different. A possible solution was proposed in the 1990s by Edward Witten [3].

1.2 M-Theory

Mathematician and physicist Edward Witten showed that the five competing superstring theories, along with the theory of 11-dimensional supergravity, were all in fact aspects of a larger, more fundamental framework, which was dubbed M-theory. It may turn out the be the unique theory of everything, which would be the holy grail of physics.

M-theory, as well as the variety of string theories, requires that the universe have extra spatial dimensions beyond the traditional three. In particular, for the predictions to be correct, the universe must possess seven extra dimensions which take the form of a compact Riemannian manifold. For these dimensions to be unobserved, they must have an exceptionally small size, on the order of Planck length [3].

Determining the precise geometry and topology of the extra dimensions is quite important, since they dictate the allowable configurations of the fundamental building blocks of M-theory. These "building blocks" may be strings or higher-dimensional objects whose equations of motion ultimately determine the physics predicted by the theory. In particular, the geometry and topology of the 7-dimensional compact space predict what types of particles (and their properties) the theory allows for. Since physicists have detailed experimental data on a number of particles, restrictions may be placed on the structure of the extra dimensions [8]. At present, it is believed that they must take the form of a compact Riemannian manifold with holonomy group contained in the exceptional Lie group G_2. "Holonomy" will be defined in the next section, and G_2 is defined precisely in section 2.2.

1.3 Holonomy Groups

Given a differentiable manifold M with Riemannian metric g, a fundamental theorem of differential geometry states that there exists a unique connection on M that is compatible with the metric [2]. This is the Levi-Civita connection, denoted ∇_g. A connection introduces the idea of parallel transport on a manifold.

Let x be a point in M, and suppose $\gamma : [0,1] \to M$ is a smooth curve in M that begins and terminates at x (i.e., $\gamma(0) = \gamma(1) = x$). An arbitrary tangent vector $v \in T_xM$ of M at x may be parallel-transported along the curve γ, which results in some vector $P_\gamma(v) \in T_xM$. In general, the parallel transport depends on the connection ∇_g, which in turn depends on the metric g. Thus, each smooth loop in M based at x gives rise to a map on the tangent space $T_x M$. These maps may be composed by composing the corresponding loops based at x, giving these maps a group structure, called the *holonomy group* of M at x. A standard result is that at all points in the connected component of a manifold, the holonomy groups of M at those points are isomorphic [6]. Thus, if we work with connected manifolds, *the* holonomy group of a manifold makes sense.

It turns out that elements of the holonomy group of a manifold are linear maps on the tangent space that preserve the metric. Thus, for an n-dimensional Riemannian manifold, the holonomy group is contained in $O(n)$. In the case of the 7-dimensional space in M-theory, the holonomy group must be contained in G_2, which is properly contained in $SO(7)$ (see section 2.2 for a definition of G_2). We say such a space is a G_2-manifold [6].

1.4 Supersymmetry and BPS States

A symmetry of a physical theory is an operation that is applied to a Lagrangian, leaving the corresponding action unchanged. Spontaneous symmetry breaking occurs when a particular solution to the equations of motion of a Lagrangian fails to possess a symmetry that is present in the Lagrangian.

Supersymmetry is the name for a class of symmetries that relate bosons to fermions and vice versa. If a theory of everything like M-theory incorporates supersymmetry, it predicts that each boson or fermion has a corresponding "superpartner." For example, because the electron is a fermion, supersymmetry predicts that there is a boson corresponding to the electron, called the "selectron." As of yet, no direct experimental evidence exists to support supersymmetry.

BPS states, named after their discoverers Bogomolny, Prasad, and Sommerfield, are solutions in M-theory that are invariant under some (but not necessarily all possible) supersymmetry operations, for example, one-quarter of them. The other symmetries are spontaneously broken. The importance of BPS states stems from the fact that the are minimum energy solutions and are thus stable. Additionally, their discovery allowed physicists to advance from perturbative to non-perturbative physics in M-theory [8].

1.5 Volume-Minimizing Cycles

Unfortunately, M-theory offers little insight in how to actually identify BPS states. However, compact cycles that minimize volume in the G_2-manifold correspond to BPS states, and finding such cycles is a somewhat more tractable problem.

The technique of calibrated geometry, formally developed about two decades ago in [4], provides a means of solving for submanifolds of $\mathbb{R}^7$ that minimize volume. Special cases of such submanifolds can be projected to volume-minimizing cycles in G_2-manifolds whose universal cover is $\mathbb{R}^7$. For example, a volume-minimizing submanifold of $\mathbb{R}^7$ that is periodic in a particular direction may be projected to a volume-minimizing cycle in the G_2-manifold $\mathbb{R}^6 \times S^1$. In fact, the problem of finding such submanifolds was proposed by Witten [11]

In this thesis I am working toward finding *associative* submanifolds of $\mathbb{R}^7$, which are 3-dimensional and volume-minimizing. Associative cycles in G_2-manifolds preserve half of the space-time supersymmetries [1]. There is a nonlinear PDE system whose solutions are precisely the associative submanifolds that are graphs of functions. In this investigation, I consider a number of special cases to the equations.

Chapter 2

The Associative Calibration

2.1 Calibrated Geometry

A calibration on a Riemannian manifold M is a closed differential p-form ϕ such that ϕ evaluated on an oriented p-tuple of tangent vectors of M is less than or equal to 1. A p-dimensional submanifold N of M is *calibrated* by ϕ if ϕ attains the value 1 on all orthonormal bases of tangent p-planes of N. The fundamental theorem of calibrated geometry states that a p-dimensional cycle calibrated by ϕ is volume minimizing in its real homology class [7]. (By "cycle" we mean it in the algebraic topology sense.) To prove this statement, let C be a p-cycle in M that is calibrated by ϕ, and suppose C' is some other p-cycle in the same real homology class as M. Since there exists a $(p+1)$-cycle D such that $\partial D = C' - C$, we have:

$$\begin{aligned} \mathrm{vol}(C') &\geq \int_{C'} \phi \\ &= \int_{C+\partial D} \phi \\ &= \int_C \varphi + \int_D d\phi \\ &= \int_C \phi \\ &= \mathrm{vol}(C), \end{aligned}$$

where we have used the fact that ϕ is less than or equal to the volume form on C', Stokes' Theorem, the fact that $d\phi = 0$, and the fact that ϕ equals the volume form on C, respectively.

In practice, it is a nontrivial matter of finding examples of calibrated submanifolds for a particular calibration on M.

2.2 The Associative Calibration

By $\mathbb{O}$ denote the octonions, the unique 8-dimensional real normed divisional algebra [9], with basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}, \mathbf{il}, \mathbf{jl}, \mathbf{kl}\}$. Let x, y, and z be elements of $\operatorname{Im}\mathbb{O}$, the set of imaginary octonions. If x, y and z form the canonically-oriented part of a quaternion subalgebra of $\mathbb{O}$, we say that the span of x, y, and z is an *associative 3-plane*. Since the quaternion subalgebras are precisely the associative 4-dimensional subalgebras of $\mathbb{O}$, this terminology makes sense.

Now, consider the following alternating trilinear form on $\operatorname{Im}\mathbb{O}$:

$$\varphi(x, y, z) = \langle x, yz \rangle$$

This form is often called the *associative form*, or the *associative calibration*. Harvey and Lawson show that φ is a constant linear combination of alternating 3-forms on $\operatorname{Im}\mathbb{O}$, so φ is closed. Also, if x, y, and z all have norm 1, then:

$$\varphi(x, y, z) = \langle x, yz \rangle \leq |x||yz| = |x||y||z| = 1,$$

so φ is indeed a calibration. A basic result of [4] is that φ equals 1 on orthonormal triples precisely when its arguments span an associative 3-plane. We now define an *associative submanifold (cycle)* of $\operatorname{Im}\mathbb{O}$ to be a 3-dimensional submanifold (cycle) calibrated by φ.

At this point it is possible to give a definition of the exceptional Lie group G_2. Often, it is defined as the group of algebra automorphisms of the octonions:

$$G_2 = \{g \in GL(\mathbb{O}) : g(xy) = g(x)g(y) \text{ for all } x, y, \in \mathbb{O}\}.$$

It is not hard to show that elements of G_2 fix all real numbers, so that we may regard G_2 as acting on $\operatorname{Im}\mathbb{O}$. Furthermore, G_2 transformations are linear and preserve the inner product, as well as orientation. Thus G_2 is isomorphic to a subgroup of $SO(7)$.

An alternative characterization of G_2 is that it is the set of orthogonal transformations of $\operatorname{Im}\mathbb{O}$ that preserve the associative form φ:

$$G_2 = \{g \in O(\operatorname{Im}\mathbb{O}) : g^*\varphi = \varphi\}.$$

This background information is detailed in [4].

2.3 Submanifolds Calibrated by the Associative Form

Let $\mathbb{H}$ denote the quaternions, the unique 4-dimensional real normed division algebra, with basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, and let $\operatorname{Im}\mathbb{H}$ denote the span of $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Let $\Omega \subset \operatorname{Im}\mathbb{H}$ be a domain and suppose $f : \Omega \to \mathbb{H}$ is smooth.

We will denote points in Ω by $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ for real quantities x, y, and z. f may be viewed as a map from a subset of $\mathbb{R}^3$ to $\mathbb{R}^4$:

$$f(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = f^1(x,y,z) + f^2(x,y,z)\mathbf{i} + f^3(x,y,z)\mathbf{j} + f^4(x,y,z)\mathbf{k}, \quad \text{or}$$
$$f(x,y,z) = \Big(f^1(x,y,z), f^2(x,y,z), f^3(x,y,z), f^4(x,y,z)\Big)$$

for smooth real-valued functions f^1, f^2, f^3, and f^4.

The graph of f is a 3-dimensional submanifold of $\operatorname{Im}\mathbb{H} \oplus \mathbb{H} \cong \mathbb{R}^7$. According to [4], the graph of f is an associative submanifold of $\mathbb{R}^7$ if and only if f satisfies the *associator equation*,

$$Df = \sigma f, \tag{2.1}$$

where

$$Df = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} - \frac{\partial f}{\partial z}\mathbf{k}, \quad \text{and} \quad \sigma f = \frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y} \times \frac{\partial f}{\partial z}.$$

The triple cross product of octonions (and quaternions) is defined below; for now it is worth mentioning that the σf term introduces substantial nonlinearities into the associator equation.

Throughout the course of this work, we will be especially interested in identifying solutions f that are periodic in one of their components, since these may be projected to the G_2-manifold $\mathbb{R}^6 \times S^1$. Ideally, a solution would, for example, determine f_1 and f_2 in terms of chosen functions f_3 and f_4, which could be periodic.

2.4 An Alternate Form of the Associator Equation

The associator equation consists of 12 unknown quantities (the three first-order partial derivatives each of f^1, f^2, f^3, and f^4). In this section we derive a convenient form in which to write the system of equations. We will often use the following subscript notation for derivatives:

$$f^i_x = \frac{\partial f^i}{\partial x}, \qquad f^i_y = \frac{\partial f^i}{\partial y}, \qquad f^i_z = \frac{\partial f^i}{\partial z},$$

for $i = 1, 2, 3, 4$. Then the Df term becomes:

$$\begin{aligned}
Df &= -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} - \frac{\partial f}{\partial z}\mathbf{k} \\
&= -\left(\frac{\partial f^1}{\partial x} + \frac{\partial f^2}{\partial x}\mathbf{i} + \frac{\partial f^3}{\partial x}\mathbf{j} + \frac{\partial f^4}{\partial x}\mathbf{k}\right)\mathbf{i} \\
&\quad - \left(\frac{\partial f^1}{\partial y} + \frac{\partial f^2}{\partial y}\mathbf{i} + \frac{\partial f^3}{\partial y}\mathbf{j} + \frac{\partial f^4}{\partial y}\mathbf{k}\right)\mathbf{j} \\
&\quad - \left(\frac{\partial f^1}{\partial z} + \frac{\partial f^2}{\partial z}\mathbf{i} + \frac{\partial f^3}{\partial z}\mathbf{j} + \frac{\partial f^4}{\partial z}\mathbf{k}\right)\mathbf{k} \\
&= \left[f_x^2 + f_y^3 + f_z^4\right] + \left[-f_x^1 + f_y^4 - f_z^3\right]\mathbf{i} \\
&\quad + \left[-f_x^4 - f_y^1 + f_z^2\right]\mathbf{j} + \left[f_x^3 - f_y^2 - f_z^1\right]\mathbf{k}.
\end{aligned}$$

Writing out the σf expression in terms of components is more complicated, but straightforward nonetheless. Using appendix IV. B. of [4] we find the definition of the triple cross product for octonions (and thus for quaternions):

$$\begin{aligned}
\sigma f &= \frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y} \times \frac{\partial f}{\partial z} \\
&= \frac{1}{2}\left[\frac{\partial f}{\partial x}\overline{\frac{\partial f}{\partial y}}\frac{\partial f}{\partial z} - \frac{\partial f}{\partial z}\overline{\frac{\partial f}{\partial y}}\frac{\partial f}{\partial x}\right].
\end{aligned}$$

Using Mathematica 4.0 to simplify this expression,[1] we arrive at:

$$\begin{aligned}
\sigma f = {} & (-f_z^2 f_y^3 f_x^4 + f_y^2 f_z^3 f_x^4 + f_z^2 f_x^3 f_y^4 - f_x^2 f_z^3 f_y^4 - f_y^2 f_x^3 f_z^4 + f_x^2 f_y^3 f_z^4) \\
& + (f_z^1 f_y^3 f_x^4 - f_y^1 f_z^3 f_x^4 - f_z^1 f_x^3 f_y^4 + f_x^1 f_z^3 f_y^4 + f_y^1 f_x^3 f_z^4 - f_x^1 f_y^3 f_z^4)\mathbf{i} \\
& + (-f_z^1 f_y^2 f_x^4 + f_y^1 f_z^2 f_x^4 + f_z^1 f_x^2 f_y^4 - f_x^1 f_z^2 f_y^4 - f_y^1 f_x^2 f_z^4 + f_x^1 f_y^2 f_z^4)\mathbf{j} \\
& + (f_z^1 f_y^2 f_x^3 - f_y^1 f_z^2 f_x^3 - f_z^1 f_x^2 f_y^3 + f_x^1 f_z^2 f_y^3 + f_y^1 f_x^2 f_z^3 - f_x^1 f_y^2 f_z^3)\mathbf{k}
\end{aligned}$$

Now, to write out $Df = \sigma f$, we must equate the four components ($1, \mathbf{i}, \mathbf{j}$, and $\mathbf{k}$). The key observation is that the components of Df look like traces of matrices, while those of σf resemble matrix determinants. In particular,

$$\begin{aligned}
Df &= \operatorname{tr} A + \operatorname{tr} B\ \mathbf{i} + \operatorname{tr} C\ \mathbf{j} + \operatorname{tr} D\ \mathbf{k} \\
\sigma f &= \det A + \det B\ \mathbf{i} + \det C\ \mathbf{j} + \det D\ \mathbf{k},
\end{aligned}$$

[1] The command `Quaternion[a,b,c,d]` is used to represent the number a+bi+cj+dk. To multiply quaternions, the noncommutative multiplication operator, `**`, must be used.

where

$$A = \begin{pmatrix} f_x^2 & f_x^3 & f_x^4 \\ f_y^2 & f_y^3 & f_y^4 \\ f_z^2 & f_z^3 & f_z^4 \end{pmatrix}, \qquad B = \begin{pmatrix} -f_x^1 & -f_x^3 & -f_x^4 \\ -f_z^1 & -f_z^3 & -f_z^4 \\ f_y^1 & f_y^3 & f_y^4 \end{pmatrix},$$

$$C = \begin{pmatrix} -f_y^1 & -f_y^2 & -f_y^4 \\ f_z^1 & f_z^2 & f_z^4 \\ -f_x^1 & -f_x^2 & -f_x^4 \end{pmatrix}, \qquad D = \begin{pmatrix} -f_z^1 & -f_z^2 & -f_z^3 \\ -f_y^1 & -f_y^2 & -f_y^3 \\ f_x^1 & f_x^2 & f_x^3 \end{pmatrix}.$$

Thus, (2.1) is equivalent to the conditions:

$$\operatorname{tr} A = \det A, \quad \operatorname{tr} B = \det B, \quad \operatorname{tr} C = \det C, \quad \operatorname{tr} D = \det D.$$

In general, the entries of the matrices A, B, C, and D are functions of x, y, and z.

2.5 The Associator Equation in Matrix Form

Alternatively, we can rearrange the four components of the associator equation as:

$$f_x^2\left(\begin{vmatrix} f_z^3 & f_z^4 \\ f_y^3 & f_y^4 \end{vmatrix} + 1\right) - f_x^3 \begin{vmatrix} f_z^2 & f_z^4 \\ f_y^2 & f_y^4 \end{vmatrix} - f_x^4 \begin{vmatrix} f_y^2 & f_y^3 \\ f_z^2 & f_z^3 \end{vmatrix} = -f_y^3 - f_z^4$$

$$f_x^1\left(\begin{vmatrix} f_y^3 & f_y^4 \\ f_z^3 & f_z^4 \end{vmatrix} - 1\right) - f_x^3 \begin{vmatrix} f_y^1 & f_y^4 \\ f_z^1 & f_z^4 \end{vmatrix} - f_x^4 \begin{vmatrix} f_z^1 & f_z^3 \\ f_y^1 & f_y^3 \end{vmatrix} = f_z^3 - f_y^4$$

$$-f_x^1 \begin{vmatrix} f_y^2 & f_y^4 \\ f_z^2 & f_z^4 \end{vmatrix} - f_x^2 \begin{vmatrix} f_z^1 & f_z^4 \\ f_y^1 & f_y^4 \end{vmatrix} + f_x^4\left(\begin{vmatrix} f_z^1 & f_z^2 \\ f_y^1 & f_y^2 \end{vmatrix} - 1\right) = f_y^1 - f_z^2$$

$$-f_x^1 \begin{vmatrix} f_z^2 & f_z^3 \\ f_y^2 & f_y^3 \end{vmatrix} - f_x^2 \begin{vmatrix} f_y^1 & f_y^3 \\ f_z^1 & f_z^3 \end{vmatrix} + f_x^3\left(\begin{vmatrix} f_y^1 & f_y^2 \\ f_z^1 & f_z^2 \end{vmatrix} + 1\right) = f_y^2 + f_z^1.$$

These four equations can be written in matrix form as $M\vec{x} = \vec{v}$, where:

$$M = \begin{bmatrix} 0 & 1 + \begin{vmatrix} f_z^3 & f_z^4 \\ f_y^3 & f_y^4 \end{vmatrix} & \begin{vmatrix} f_y^2 & f_y^4 \\ f_z^2 & f_z^4 \end{vmatrix} & -\begin{vmatrix} f_y^2 & f_y^3 \\ f_z^2 & f_z^3 \end{vmatrix} \\ -1 - \begin{vmatrix} f_z^3 & f_z^4 \\ f_y^3 & f_y^4 \end{vmatrix} & 0 & -\begin{vmatrix} f_y^1 & f_y^4 \\ f_z^1 & f_z^4 \end{vmatrix} & \begin{vmatrix} f_y^1 & f_y^3 \\ f_z^1 & f_z^3 \end{vmatrix} \\ -\begin{vmatrix} f_y^2 & f_y^4 \\ f_z^2 & f_z^4 \end{vmatrix} & \begin{vmatrix} f_y^1 & f_y^4 \\ f_z^1 & f_z^4 \end{vmatrix} & 0 & -\begin{vmatrix} f_y^1 & f_y^2 \\ f_z^1 & f_z^2 \end{vmatrix} - 1 \\ \begin{vmatrix} f_y^2 & f_y^3 \\ f_z^2 & f_z^3 \end{vmatrix} & -\begin{vmatrix} f_y^1 & f_y^3 \\ f_z^1 & f_z^3 \end{vmatrix} & \begin{vmatrix} f_y^1 & f_y^2 \\ f_z^1 & f_z^2 \end{vmatrix} + 1 & 0 \end{bmatrix} \tag{2.2}$$

and

$$\vec{x} = \begin{bmatrix} f_x^1 \\ f_x^2 \\ f_x^3 \\ f_x^4 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} -f_y^3 - f_z^4 \\ f_z^3 - f_y^4 \\ f_y^1 - f_z^2 \\ f_y^2 + f_z^1 \end{bmatrix}.$$

It is important to keep in mind that these quantities are functions of x, y, and z. Note that M is skew-symmetric.

As a side note this formulation allows us a convenient method of classifying "most" of the associative 3-planes in $\mathbb{R}^7$:

Theorem 1. *Suppose the graph of $f : \mathbb{R}^3 \to \mathbb{R}^4$ is a 3-dimensional plane P in $\mathbb{R}^7$. The components f^i, $i = 1, 2, 3, 4$, of f take the form:*

$$f^i(x, y, z) = a_i x + b_i y + c_i z + d_i$$

for real constants a_i, b_i, c_i, and d_i. Then P is associative if and only if a_i, b_i, and c_i satisfy the above equation. In the case that M is invertible, it may be inverted to solve for the a_i in terms of the other constants.

Remark: As objects in $\mathbb{R}^7$, these linear solutions are not geometrically interesting *per se*. However, they may possibly be projected to nontrivial volume-minimizing cycles in, for example, the G_2-manifold $\mathbb{R}^6 \times S^1$.

2.6 Previous Research

In 2001 Ian Weiner [10] found examples of associative submanifolds of $\mathbb{R}^7$ by studying graphs of functions $f : \operatorname{Im}\mathbb{H} \to \mathbb{H}$ that are invariant under a particular 1-parameter subgroup of G_2. No periodic examples were found.

Next, Matthew Holden [5] considered solutions of the form $Df = \sigma f = 0$, which occurs when the partial derivatives of f are linearly dependent. Interesting examples of periodic associative submanifolds were given.

Chapter 3

$Df = \sigma f =$ Nonzero Constant

Matthew Holden solved for a particular class of functions $f : \mathbb{R}^3 \to \mathbb{R}^4$ satisfying $Df = \sigma f = 0$. In what follows the case in which $Df = \sigma f = r$, for $r \in \mathbb{R}, r \neq 0$ is considered. Some results that place limitations on possible solutions are proven.

3.1 Preliminary Results

In order to simplify the notation, we introduce the following substitutions:

$$a_i = \frac{\partial f^i}{\partial x} \qquad b_i = \frac{\partial f^i}{\partial y} \qquad c_i = \frac{\partial f^i}{\partial z},$$

where a_i, b_i, and c_i are functions of x, y, z. Our first result is:

Theorem 2. *Suppose f satisfies $\sigma f = r$, where r a real, nonzero constant. Then the real component of f is a constant function.*

This theorem is true even if r is allowed to depend on x, y, and z, but for now we are only interested in the case in which r is a constant.

Proof. The fact that $\sigma f = r \neq 0$ is equivalent to $\det A = r$ and $\det B = \det C = \det D = 0$. Our approach is to show that either $a_1 = b_1 = c_1 = 0$ or $\det A = 0$, where the latter leads to a contradiction.

Recall the rows of a matrix of determinant zero must be linearly dependent. Thus, there exist smooth functions β_i, γ_i, and δ_i, $i = 1, 2, 3$, such that:

$$\beta_1 \begin{pmatrix} a_1 \\ a_3 \\ a_4 \end{pmatrix} + \beta_2 \begin{pmatrix} b_1 \\ b_3 \\ b_4 \end{pmatrix} + \beta_3 \begin{pmatrix} c_1 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.1}$$

$$\gamma_1 \begin{pmatrix} a_1 \\ a_2 \\ a_4 \end{pmatrix} + \gamma_2 \begin{pmatrix} b_1 \\ b_2 \\ b_4 \end{pmatrix} + \gamma_3 \begin{pmatrix} c_1 \\ c_2 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.2}$$

$$\delta_1 \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \delta_2 \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \delta_3 \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.3}$$

with neither the β_i's, γ_i's, nor δ_i's simultaneously zero. Consider the first component of each of these three vector equations. Together, these form the matrix equation:

$$\begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ \delta_1 & \delta_2 & \delta_3 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.4}$$

Let the matrix on the left be called Δ. Assume, by way of contradiction, that $\det \Delta = 0$. Then the rows of Δ are linearly dependent, so there exist smooth functions k_1, k_2, and k_3 not simultaneously zero such that:

$$k_1 \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} + k_2 \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} + k_3 \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.5}$$

At any given point (x, y, z), at least one of k_1, k_2, and k_3 is nonzero. In the event that $k_1 \neq 0$, we can solve for each β_i in terms of the other quantities:

$$\beta_i = -\frac{k_2}{k_1}\gamma_i - \frac{k_3}{k_1}\delta_i. \tag{3.6}$$

Multiplying (3.2) by $-\frac{k_2}{k_1}$, (3.3) by $-\frac{k_3}{k_1}$, and adding yields:

$$-\frac{k_2}{k_1}\gamma_1 \begin{pmatrix} a_1 \\ a_2 \\ a_4 \end{pmatrix} - \frac{k_3}{k_1}\delta_1 \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} - \frac{k_2}{k_1}\gamma_2 \begin{pmatrix} b_1 \\ b_2 \\ b_4 \end{pmatrix} - \frac{k_3}{k_1}\delta_2 \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \dots$$

$$\dots - \frac{k_2}{k_1}\gamma_3 \begin{pmatrix} c_1 \\ c_2 \\ c_4 \end{pmatrix} - \frac{k_3}{k_1}\delta_3 \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.7}$$

Using (3.6), the middle component of this equation reduces to:

$$\beta_1 a_2 + \beta_2 b_2 + \beta_3 c_2 = 0.$$

Together with the second two components of (3.1), we now have:

$$\begin{pmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since β_1, β_2, and β_3 are not all zero, there is a nontrivial solution to the homogeneous equation $A^T\vec{x} = \vec{0}$, so $\det A^T = \det A = 0$. This contradicts the fact that $\det A = r \neq 0$.

The next case to consider is that for $k_1 = 0$. Suppose first that $k_2 \neq 0$. Then (3.5) can be solved for γ_i:

$$\gamma_i = -\frac{k_3}{k_2}\delta_i.$$

Multiply (3.3) by $-\frac{k_3}{k_2}$ to obtain:

$$-\frac{k_3}{k_2}\delta_1 \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} - \frac{k_3}{k_2}\delta_2 \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - \frac{k_3}{k_2}\delta_3 \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Substituting $\gamma_i = -\frac{k_3}{k_2}\delta_i$, the third component of the last equation becomes:

$$\gamma_1 a_3 + \gamma_2 b_3 + \gamma_3 c_3 = 0.$$

Together with (3.2), we have:

$$\begin{pmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

But γ_1, γ_2, and γ_3 are never simultaneously zero, so $\det A^T = \det A = 0$. This contradicts the assumption that $\det A = r \neq 0$.

The last case to consider is $k_1 = k_2 = 0$, but $k_3 \neq 0$. Then (3.5) becomes:

$$k_3 \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

But $k_3 \neq 0$, and the δ_i's are never simultaneously zero, so this case also leads to a contradiction.

All cases of $\det \Delta = 0$ led to contradictions, so $\det \Delta \neq 0$. From (3.4), it follows that $a_1 = b_1 = c_1 = 0$. Then the first order partial derivatives of f^1 are all zero, so f^1 is a constant function. □

Remark: Theorem 4.6 of [5] states that solutions to $Df = q$ for a constant quaternion q have harmonic components. Thus, in our case, f^2, f^3, and f^4 are harmonic on Ω.

Corollary 1. *For a nonzero real constant r, we have the following:*

(i) $\sigma f = r\mathbf{i}$ *implies* f^2 *is constant;*

(ii) $\sigma f = r\mathbf{j}$ *implies* f^3 *is constant;*

(iii) $\sigma f = r\mathbf{k}$ *implies* f^4 *is constant.*

Proof: We prove only (i), since the other cases follow in the same manner. Suppose f is a solution to $\sigma f = r\mathbf{i}$. Then $-\mathbf{i}\,\sigma f = r$. It will now be shown that $\sigma(-\mathbf{i}f) = -\mathbf{i}\,\sigma f$:

$$\begin{aligned}
\sigma(-\mathbf{i}f) &= \frac{1}{2}\left[\left(-\mathbf{i}\frac{\partial f}{\partial x}\right)\left(\overline{-\mathbf{i}\frac{\partial f}{\partial y}}\right)\left(-\mathbf{i}\frac{\partial f}{\partial z}\right) - \left(-\mathbf{i}\frac{\partial f}{\partial z}\right)\left(\overline{-\mathbf{i}\frac{\partial f}{\partial y}}\right)\left(-\mathbf{i}\frac{\partial f}{\partial x}\right)\right] \\
&= \frac{1}{2}\left[\left(-\mathbf{i}\frac{\partial f}{\partial x}\right)\left(\overline{\frac{\partial f}{\partial y}}\mathbf{i}\right)\left(-\mathbf{i}\frac{\partial f}{\partial z}\right) - \left(-\mathbf{i}\frac{\partial f}{\partial z}\right)\left(\overline{\frac{\partial f}{\partial y}}\mathbf{i}\right)\left(-\mathbf{i}\frac{\partial f}{\partial x}\right)\right] \\
&= -\mathbf{i}\,\frac{1}{2}\left[\left(\frac{\partial f}{\partial x}\right)\left(\overline{\frac{\partial f}{\partial y}}\right)\left(\frac{\partial f}{\partial z}\right) - \left(\frac{\partial f}{\partial z}\right)\left(\overline{\frac{\partial f}{\partial y}}\right)\left(\frac{\partial f}{\partial x}\right)\right] \\
&= -\mathbf{i}\,\sigma f.
\end{aligned}$$

Thus $\sigma(-\mathbf{i}f) = r$. By Theorem 2, the real component of $-\mathbf{i}f$ is constant. But:

$$\begin{aligned}
-\mathbf{i}f &= -\mathbf{i}(f^1 + f^2\mathbf{i} + f^3\mathbf{j} + f^4\mathbf{k}) \\
&= f^2 - f^1\mathbf{i} - f^3\mathbf{k} + f^4\mathbf{j},
\end{aligned}$$

so f^2 is constant. □

Remark: We should not expect to have a similar result for $\sigma f = r + s\mathbf{i}$, a complex constant, for $r, s \neq 0$. The above proof crucially depended on all of $\det B$, $\det C$, and $\det D$ equalling zero, but $\sigma f = r + s\mathbf{i}$ requires $\det B \neq 0$.

Corollary 2. *Suppose f is a solution to $Df = \sigma f = r$ for a nonzero real constant r. Then the matrix:*

$$A = \begin{pmatrix} \frac{\partial f^2}{\partial x} & \frac{\partial f^3}{\partial x} & \frac{\partial f^4}{\partial x} \\ \frac{\partial f^2}{\partial y} & \frac{\partial f^3}{\partial y} & \frac{\partial f^4}{\partial y} \\ \frac{\partial f^2}{\partial z} & \frac{\partial f^3}{\partial z} & \frac{\partial f^4}{\partial z} \end{pmatrix}$$

is symmetric.

Proof: First, recall the notation from Chapter 2:

$$A = \begin{pmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{pmatrix}.$$

Since $Df = r$, it follows,

$$Df = \operatorname{tr} A + \operatorname{tr} B\mathbf{i} + \operatorname{tr} C\mathbf{j} + \operatorname{tr} D\mathbf{k} = r,$$

so the traces of B, C, and D are zero. Then:

$$\begin{aligned} \operatorname{tr} B &= -a_1 - c_3 + b_4 = 0 \\ \operatorname{tr} C &= -b_1 + c_2 - a_4 = 0 \\ \operatorname{tr} D &= -c_1 - b_2 + a_3 = 0. \end{aligned}$$

By Theorem 2, $a_1 = b_1 = c_1 = 0$, so the above equations tell us $c_3 = b_4, c_2 = a_4$, and $b_2 = a_3$. Thus, A is symmetric. □

In the following sections, we will use the above results to investigate specials cases of $Df = \sigma f = r$.

3.2 Case 1: Diagonal A

Suppose that A takes the form:

$$A = \begin{pmatrix} a_2 & 0 & 0 \\ 0 & b_3 & 0 \\ 0 & 0 & c_4 \end{pmatrix} = \begin{pmatrix} \frac{\partial f^2}{\partial x} & \frac{\partial f^3}{\partial x} & \frac{\partial f^4}{\partial x} \\ \frac{\partial f^2}{\partial y} & \frac{\partial f^3}{\partial y} & \frac{\partial f^4}{\partial y} \\ \frac{\partial f^2}{\partial z} & \frac{\partial f^3}{\partial z} & \frac{\partial f^4}{\partial z} \end{pmatrix}.$$

This immediately restricts the possible dependencies of f^2, f^3, and f^4 to $f^2 = f^2(x), f^3 = f^3(y)$, and $f^4 = f^4(z)$. It follows that $a_2 = \frac{\partial f^2}{\partial x}$ is a function of only x, and similarly, b_3 and c_4 are functions of only y and z, respectively.

By the requirement $\operatorname{tr} A = r$, we have that $a_2 + b_3 + c_4 = r$. Then

$$r - a_2(x) = b_3(y) + c_4(z).$$

Since the left hand side has only x dependence and the right hand side has no x dependence, both sides of the equation equal some constant k_1. In particular, this implies $a_2(x) = a_2$ is a constant. Rearranging yields

$$r - k_1 - b_3(y) = c_4(z),$$

and by a similar argument, both sides equal some constant k_2. In particular, b_3 and c_4 are constants. Then a necessary set of conditions for $f = (f^1, f^2, f^3, f^4)$ to be a solution to $Df = \sigma f = r$ is that:

$$\begin{aligned} f^2(x,y,z) &= a_2 x + a_0 \\ f^3(x,y,z) &= b_3 y + b_0 \\ f^4(x,y,z) &= c_4 z + c_0. \end{aligned}$$

The graphs of such solutions must correspond to special cases of 3-planes. However, this is not surprising, considering the highly restrictive assumption that was placed on A.

Given that solutions in this case are linear, we may proceed to explicitly classify all such solutions. The relevant parameters are a_2, b_3, c_4, and r, with the two constraints $\operatorname{tr} A = a_2 + b_3 + c_4 = r$ and $\det A = a_2 b_3 c_4 = r$. Thus, we expect two free parameters, say a_2 and b_3. To determine c_4, start with $\operatorname{tr} A = \det A$:

$$\begin{aligned} a_2 + b_3 + c_4 &= a_2 b_3 c_4 \\ c_4 - a_2 b_3 c_4 &= -a_2 - b_3 \\ c_4 &= \frac{-a_2 - b_3}{1 - a_2 b_3}. \end{aligned}$$

Note that if $a_2 b_3 = 1$, Then $a_2 b_3 c_4 = a_2 + b_3 + c_4$ implies that $a_2 + b_3 = 0$. So $(a_2)^2 = -1$, a contradiction.

Next, we determine r in terms of the free parameters:

$$\begin{aligned} r &= a_2 b_3 c_4 \\ &= (a_2 b_3) \frac{-a_2 - b_3}{1 - a_2 b_3}. \end{aligned}$$

We have proved the following statement:

Proposition 1. *Suppose f satisfies $Df = \sigma f = r \neq 0$, such that the matrix A is diagonal. Then the components of f are given by:*

$$\begin{aligned} f^1 &= const. \\ f^2 &= a_2 x + const. \\ f^3 &= b_3 y + const. \\ f^4 &= \frac{-a_2 - b_3}{1 - a_2 b_3} z + const., \end{aligned}$$

where a_2 and b_3 are nonzero real constants with $a_2 \neq -b_3$ and $a_2b_3 \neq 1$. The constant r is given by:

$$r = (a_2b_3)\frac{-a_2 - b_3}{1 - a_2b_3}.$$

3.3 Case 2: Four zero entries

Suppose that A takes the form:

$$A = \begin{pmatrix} a_2 & 0 & a_4 \\ 0 & b_3 & 0 \\ c_2 & 0 & c_4 \end{pmatrix}, \tag{3.8}$$

with $c_2 = a_4$. Then $f^2 = f^2(x,z), f^3 = f^3(y)$, and $f^4 = f^4(x,z)$. It follows that $a_2 = a_2(x,z), b_3 = b_3(y)$, and $c_4 = c_4(x,z)$. By the condition $\operatorname{tr} A = r$, we have $a_2 + b_3 + c_4 = r$, or

$$a_2(x,z) + c_4(x,z) = r - b_3(y),$$

which immediately implies that both sides equal a constant, k. In particular, $f^3(x,y,z) = (r-k)y + b_0$. Let us now determine f^2 in terms of f^4. Since $\frac{\partial f^2}{\partial z} = \frac{\partial f^4}{\partial x}$, we have:

$$f^2(x,z) = \int \frac{\partial f^4}{\partial x} dz + g(x),$$

for some function $g(x)$. Differentiate this expression with respect to x to determine $g(x)$:

$$\frac{\partial f^2}{\partial x} = \int \frac{\partial^2 f^4}{\partial x^2} dz + g'(x).$$

But f^4 must be harmonic, so $\frac{\partial^2 f^4}{\partial x^2} = -\frac{\partial^2 f^4}{\partial z^2}$, and

$$\frac{\partial f^2}{\partial x} = -\frac{\partial f^4}{\partial z} + g'(x).$$

Thus $g'(x) = k$, which implies $g(x) = kx + g_0$ for some real constant g_0.

Now, consider the constraint $\det A = r$:

$$a_2b_3c_4 - a_4c_2b_3 = r.$$

If $b_3 = 0$, then $r = 0$, which is a contradiction. Divide out $b_3 = r - k$, and substitute $c_2 = a_4$ and $a_2 = k - c_4$ to obtain

$$(k - c_4)c_4 - (a_4)^2 = \frac{r}{r-k}.$$

If $f^4(x,z)$ is a harmonic function satisfying the above, then $f^1 + f^2\mathbf{i} + f^3\mathbf{j} + f^4\mathbf{k}$ satisfies $Df = \sigma f = r$, where f^1 is a constant, and f^2 and f^3 are as given. Thus, the problem is reduced to finding a solution f^4 to:

$$\left(k - \frac{\partial f^4}{\partial z}\right)\frac{\partial f^4}{\partial z} - \left(\frac{\partial f^4}{\partial x}\right)^2 = \frac{r}{r-k} \tag{3.9}$$

It remains an open conjecture as to whether there are any harmonic, nonlinear solutions to (3.9). It may be possible to use the method of characteristics for first-order nonlinear PDEs to answer this question. However, we can at least classify the linear solutions.

For A of the form (3.8), a solution f is described by real constants a_2, c_2, b_3, a_4, c_4, and r. We have the three constraints $c_2 = a_4$, $\operatorname{tr} A = a_2 + b_3 + c_4 = r$, and $\det A = a_2 b_3 c_4 - a_4 c_2 b_3 = r$. Let us determine the parameters in terms of a_2, b_3, and r. c_2 is equal to a_4, and c_4 is readily determined from $a_2 + b_3 + c_4 = r$. Consider $\det A = r$. Since $b_3 = 0$ implies that $\det A = 0$ by (3.8), we may assume $b_3 \neq 0$. Then we have:

$$\begin{aligned} a_2 c_4 - a_4 c_2 &= \frac{r}{b_3} \\ a_2 c_4 - (a_4)^2 &= \frac{r}{b_3} \\ a_2(r - a_2 - b_3) - (a_4)^2 &= \frac{r}{b_3}, \quad \text{or} \\ a_4 &= \pm\sqrt{a_2(r - a_2 - b_3) - \frac{r}{b_3}}, \end{aligned}$$

provided the square root exists. We have proved:

Proposition 2. *Suppose f satisfies $Df = \sigma f = r \neq 0$, such that the matrix A satisfies (3.8). Then the components of f are given by:*

$$\begin{aligned} f^1 &= const. \\ f^2 &= a_2 x + \pm\sqrt{a_2(r - a_2 - b_3) - \frac{r}{b_3}}\, z + const. \\ f^3 &= b_3 y + const. \\ f^4 &= \pm\sqrt{a_2(r - a_2 - b_3) - \frac{r}{b_3}}\, x + (r - a_2 - b_3) z + const., \end{aligned}$$

provided the square root exists, where $b_3 \neq 0, r \neq 0$, and a_2 are real constants.

3.4 Case 3A: Three zero entries

Consider the case:

$$A = \begin{pmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & 0 \\ c_2 & 0 & 0 \end{pmatrix}, \tag{3.10}$$

where $c_2 = a_4$ and $b_2 = a_3$. Since several of the partial derivatives are zero, we can conclude $f^2 = f^2(x,y,z)$, $f^3 = f^3(x,y)$, and $f^4 = f^4(x)$. Let's begin by applying the symmetry condition $a_4 = c_2$. $a_4 = \frac{\partial f^4}{\partial x}$ is a function of x alone, so:

$$\begin{aligned} f^2(x,y,z) &= \int \frac{\partial f^2}{\partial z} dz + g(x,y) = \int c_2 dz + g(x,y) \\ &= \int a_4 dz + g(x,y) = a_4(x)z + g(x,y), \end{aligned}$$

for some function $g(x,y)$. This knowledge about f^2 allows us to compute a_2:

$$a_2 = \frac{\partial f^2}{\partial x} = a_4'(x)z + \frac{\partial g}{\partial x}.$$

But the condition $\operatorname{tr} A = r$ dictates that $a_2 = b_3 - r$. Then:

$$a_4'(x)z + \frac{\partial g}{\partial x}(x,y) = r - b_3(x,y).$$

To account for the factor of z on the left hand side, we must have $a_4'(x) = 0$, so that a_4 is a constant. Then by symmetry of A, c_2 is constant.

Now, the condition $\det A = r$ requires that $-a_4 b_3 c_2 = r$. Since r must be nonzero, and a_4 and c_2 are constants, it follows that b_3 is a constant as well.

Since a_2 and c_2 are constants, f^2 must take the form:

$$f^2(x,y,z) = a_2 x + c_2 z + \int b_2(y) dy.$$

Similarly, since b_3 is a constant, f^3 takes the form:

$$f^3(x,y) = \int a_3(x) dx + b_3 y.$$

Then the symmetry requirement $b_2 = a_3$ implies $b_2(y) = a_3(x)$, so that b_2 and a_3 are constants.

We have shown that all the partial derivatives of f^2, f^3, and f^4 are constants, so that solutions of this form have graphs that are 3-planes. We proceed to characterize these solutions. The parameters of interest are $a_2, b_2, c_2, a_3, b_3, a_4$, and r, and we have constraints $c_2 = a_4$, $b_2 = a_3$, $a_2 + b_3 = r$, and $-a_4 b_3 c_2 = r$. By substitution into the last constraint,

$$\begin{aligned} -(a_4)^2 b_3 &= r \\ -(a_4)^2 (r - a_2) &= r, \quad \text{or} \\ a_2 &= \frac{r}{(a_4)^2} + r. \end{aligned}$$

Note that a_4 cannot equal zero by (3.10), since $\det A$ would then equal zero. The last parameter to be determined is:

$$\begin{aligned} b_3 &= r - a_2 \\ &= -\frac{r}{(a_4)^2}, \end{aligned}$$

and we have proved:

Proposition 3. *Suppose that f is a solution to $Df = \sigma f = r \neq 0$, such that the matrix A satisfies (3.10). Then the components of f are given by:*

$$\begin{aligned} f^1 &= const. \\ f^2 &= \left(\frac{r}{(a_4)^2} + r \right) x + b_2 y + a_4 z + const. \\ f^3 &= b_2 x - \frac{r}{(a_4)^2} y + const. \\ f^4 &= a_4 z + const., \end{aligned}$$

where $r \neq 0, a_4 \neq 0$, and b_3 are real constants.

3.5 Case 3B: Three zero entries

Next, consider the somewhat similar case:

$$A = \begin{pmatrix} a_2 & a_3 & 0 \\ b_2 & 0 & b_4 \\ 0 & c_3 & c_4 \end{pmatrix}, \tag{3.11}$$

with $b_2 = a_3$ and $c_3 = b_4$. Then $f^2 = f^2(x,y), f^3 = f^3(x,z)$, and $f^4 = f^4(y,z)$. It follows that a_2 is a function of x and y only, while c_4 is a function of y and z only. The equation $\operatorname{tr} A = a_2 + c_4 - r$ implies that $a_2 = \frac{\partial f^2}{\partial x}$ may not depend on x, and $c_4 = \frac{\partial f^4}{\partial z}$ may not depend on z. Next, the equation $\det A = r = -a_2 b_4^2 - a_3^2 c_4$ may be separated as:

$$-a_2 b_4^2 = r + a_3^2 c_4.$$

The only possible x-dependence in this equation is from a_3; thus $a_3 = \frac{\partial f^3}{\partial x}$ may not depend on x. All of this information restricts f^2, f^3, and f^4 to the following forms:

$$\begin{aligned} f^2(x,y) &= a_2(y)x + \alpha(y) \\ f^3(x,z) &= a_3(z)x + \beta(z) \\ f^4(y,z) &= c_4(y)z + \gamma(y), \end{aligned}$$

for some functions $\alpha(y), \beta(z)$, and $\gamma(y)$.

By the symmetry of the matrix A, we have $c_3 = b_4$ and $b_2 = a_3$. We proceed to examine these constraints one-by-one in terms of the above functional forms of f^2, f^3, and f^4:

$$\begin{aligned} c_3 &= \frac{\partial f^3}{\partial z} = a_3'(z)x + \beta'(z), \\ b_4 &= \frac{\partial f^4}{\partial y} = c_4'(y)z + \gamma'(y). \end{aligned}$$

If we set $c_3 = b_4$, the free x term forces $a_3'(z) = 0$, so that a_3 is a constant. Since symmetry of A dictates that $a_3 = b_2$, we have that b_2 is a constant as well. Also,

$$\begin{aligned} b_2 &= \frac{\partial f^2}{\partial y} = a_2'(y)x + \alpha'(y), \\ a_3 &= \frac{\partial f^3}{\partial x} = a_3. \end{aligned}$$

Setting these expressions equal, we see that $a_2'(y)$ must be zero, so that a_2 is a constant. By the trace equation $a_2 + c_4 = r$, it follows that c_4 is a constant. Now c_3 reduces to a function of z alone, while b_4 is a function of y alone. To ensure symmetry of A, $c_3 = b_4$, so that both are constants. We have established that all the partial derivatives of f^2, f^3, and f^4 are constants. Let us proceed to explicitly determine the class of such solutions.

Recall that $b_2 = a_3, c_3 = b_4, c_4 = r - a_2$, and $r = -a_2(b_4)^2 - (a_3)^2 c_4$. By substitution, the latter becomes:

$$r = -a_2(b_4)^2 - (a_3)^2(r - a_2).$$

Now we have:

Proposition 4. *Suppose f is a solution to $Df = \sigma f = r \neq 0$, such that the matrix A satisfies (3.11). Then the components of f are given by:*

$$\begin{aligned} f^1 &= const. \\ f^2 &= a_2 x + a_3 y + const. \\ f^3 &= a_3 x + b_4 z + const. \\ f^4 &= b_4 y + (r - a_2) z + const., \end{aligned}$$

where a_2, b_4, and a_3 are real constants such that the number

$$r = -a_2 (b_4)^2 - (a_3)^2 (r - a_2)$$

is nonzero.

3.6 Case 4A: Two zero entries on the diagonal

Suppose we would like to consider the case in which A has precisely two identically zero entries. These two entries must either both lie on the diagonal of A or must be symmetrically off-diagonal. In this section, we consider the former. Suppose A takes the form:

$$A = \begin{pmatrix} a_2 & a_3 & a_4 \\ b_2 & 0 & b_4 \\ c_2 & c_3 & 0 \end{pmatrix}, \tag{3.12}$$

where $b_2 = a_3$, $c_2 = a_4$, and $c_3 = b_4$. Since $\operatorname{tr} A = r$, a_2 must equal the constant r. Then f^2 takes the form:

$$f^2(x, y, z) = rx + \alpha(y, z),$$

for some function $\alpha(y, z)$. Also, since $b_3 = c_4 = 0$, we have that $f^3 = f^3(x, z)$ and $f^4 = f^4(x, y)$.

Now, consider the symmetry condition $b_2 = a_3$. This is equivalent to:

$$\frac{\partial \alpha}{\partial y}(y, z) = a_3(x, z).$$

Differentiating this equation with respect to x or y, we find that:

$$\frac{\partial a_3}{\partial x} = 0, \qquad \text{and} \qquad \frac{\partial^2 \alpha}{\partial y^2} = 0,$$

which implies $a_3 = a_3(z)$, and $\frac{\partial \alpha}{\partial y}$ is a function of z only. Then integration shows that $\alpha(y,z) = \alpha_1(z)y + \alpha_2(z)$ for some functions $\alpha_1(z)$ and $\alpha_2(z)$. Thus, f^2 takes the general form:

$$f^2(x,y,z) = rx + \alpha_1(z)y + \alpha_2(z).$$

Next, consider the second symmetry condition $c_2 = a_4$. In terms of derivatives, this means:

$$\alpha_1'(z)y + \alpha_2'(z) = a_4(x,y).$$

Differentiating with respect to x, we find that $\frac{\partial a_4}{\partial x} = 0$, so $a_4 = a_4(z)$. Differentiating with respect to z, we have:

$$\alpha_1''(z)y + \alpha_2''(z) = 0.$$

Since y is a free parameter, it must be that $\alpha_1''(z) = 0$, and so $\alpha_2''(z) = 0$ as well. Thus,

$$\alpha_1(z) = Kz + L, \qquad \text{and } \alpha_2(z) = Mz + N,$$

for some real constants K, L, M, and N. Then the form of f^2 is completely determined:

$$\begin{aligned} f^2(x,y,z) &= rx + (Kz + L)y + Mz + N \\ &= Kyz + rx + Ly + Mz + N. \end{aligned}$$

By the last symmetry condition, $c_3(x,z) = b_4(x,y)$. From this, we see that $c_3 = c_3(x)$ and $b_4 = b_4(x)$. Now, we shall determine the form of $f^3(x,z)$ by considering what we know about its partial derivatives:

$$\begin{aligned} \frac{\partial f^3}{\partial x} &= a_3(z) = b_2(z) = Kz + L \\ \frac{\partial f^3}{\partial x} &= c_3(x) = b_4(x). \end{aligned}$$

By integrating the first of these equations, we see $f^3(x,z) = (Kz + L)x + g(z)$ for some function $g(z)$. But $\frac{\partial f^3}{\partial y} = c_3 = b_4(x)$, so

$$Kx + g'(z) = b_4(x).$$

By the separation of variables argument, we have that $g'(z)$ is a constant, so $g(z) = Pz + Q$ for some real constants P and Q. Thus, up to undetermined constants, we know the the form of f^3:

$$\begin{aligned} f^3(x,z) &= (Kz + L)x + Pz + Q \\ &= Kxz + Lx + Pz + Q. \end{aligned}$$

Finally, to determine f^4, we can make use of the symmetry conditions, along with the known expressions for f^2 and f^3 to assist in computing derivatives:

$$\begin{aligned} \frac{\partial f^4}{\partial x} &= a_4 = c_2 = Ky + M \\ \frac{\partial f^4}{\partial y} &= b_4 = c_3 = Kx + P. \end{aligned}$$

By integrating the first of these with respect to x and the second with respect to z, we have two expressions for f^4:

$$\begin{aligned} f^4(x, y) &= (Ky + M)x + \varphi(y) \\ f^4(x, y) &= (Kx + P)y + \psi(x). \end{aligned}$$

To ensure that these expressions are consistent, subtract them and rearrange:

$$\begin{aligned} Kxy + Mx + \varphi(y) - Kxy - Py - \psi(x) &= 0, \\ Mx - \psi(x) &= Py - \varphi(y). \end{aligned}$$

Both sides must be equal to some constant R. In particular, $\varphi(y) = Py + R$. At last, f^4 is determined:

$$\begin{aligned} f^4(x, y) &= (Ky + M)x + Py + R \\ &= Kxy + Mx + Py + R. \end{aligned}$$

In the case currently under consideration, the solution to $Df = \sigma f = r$ is nonlinear precisely when $K \neq 0$. However, we will now demonstrate that the condition $\det A = r$ forces $K = 0$. We compute $\det A$ explicitly:

$$\begin{aligned} \det A &= -a_2 b_4^2 + a_3 b_4 c_2 + a_4 b_2 c_3 \\ &= -r(Kx + P)^2 + (Kz + L)(Kx + P)(Ky + M) \\ &\quad + (Ky + M)(Kz + L)(Kx + P). \end{aligned}$$

Expanding and combining like terms leads to a coefficient of $2K^3$ on the xyz term. Since x, y, and z are free parameters, it must be that $K = 0$, so that all solutions in this case are linear. Let us now precisely determine these solutions.

We have the conditions $b_2 = a_3, c_2 = a_4, c_3 = b_4$, and $a_2 = r$, along with the determinant condition:

$$
\begin{aligned}
r &= -a_2(b_4)^2 + a_3b_4c_2 + a_4b_2c_3 \\
&= -r(b_4)^2 + a_3b_4c_2 + a_4b_2c_3, \quad \text{or} \\
r(1+(b_4)^2) &= a_3b_4c_2 + a_4b_2c_3, \quad \text{or} \\
r &= \frac{a_3b_4c_2 + a_4b_2c_3}{1+(b_4)^2} \\
&= \frac{2a_3a_4c_3}{1+(b_4)^2}.
\end{aligned}
$$

The parameters a_3, c_3, and a_4 are free, provided they are nonzero. So we have:

Proposition 5. *Suppose f is a solution to $Df = \sigma f = r \neq 0$, such that the matrix A satisfies (3.12). Then the components of f are given by:*

$$
\begin{aligned}
f^1 &= const. \\
f^2 &= \left(\frac{2a_3a_4c_3}{1+(b_4)^2}\right)x + a_3y + a_4z + const. \\
f^3 &= a_3x + c_3z + const. \\
f^4 &= a_4x + c_3y + const.,
\end{aligned}
$$

where a_3, a_4, and c_3 are nonzero real constants.

3.7 Case 4B: Two zero entries off the diagonal

The last case we consider here is:

$$
A = \begin{pmatrix} a_2 & a_3 & 0 \\ b_2 & b_3 & b_4 \\ 0 & c_3 & c_4 \end{pmatrix}.
$$

To make the problem more tractable, I proceeded under the assumption $\frac{\partial^2 f^3}{\partial y^2} = 0$. By working in a similar fashion to the previous examples in this chapter, one can prove that all such solutions are indeed linear.

3.8 Diagonalization of A

Recall that real symmetric matrices are orthogonally diagonalizable over the reals. It follows that solutions with $Df = \sigma f = r$ must have a matrix A that satisfies:

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = P^T A P,$$

for smooth $\lambda_1(x,y,z), \lambda_2(x,y,z), \lambda_3(x,y,z)$, and a 3×3 matrix P whose entries are smooth functions from $\mathbb{R}^3$ to $\mathbb{R}$. Note that $\det A = \operatorname{tr} A = r$ is equivalent to:

$$\lambda_1 \lambda_2 \lambda_3 = \lambda_1 + \lambda_2 + \lambda_3 = r.$$

However, this approach has led to no interesting results to date, and is mentioned only as a suggestion for future work.

Chapter 4

The Complex Structure Approach

In this chapter we exploit the inherent quaternionic structure of (2.1), by assuming that M from (2.2) is a complex structure. A complex structure is an $n \times n$ matrix X such that $X^2 = -I_{n\times n}$. In this case, M may be decomposed as a linear combination of matrix representations of the unit quaternions $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$. The standard formula for inverting quaternions may then be used to invert M.

4.1 The Setup

Recall from section 2.5 that (2.1) may be written out in terms of partial derivatives as the matrix equation $M\vec{a} = \vec{v}$. In this chapter, we make some adjustments to M which do not change its definition. Specifically, the 2×2 determinants are swapped with their transposes, and some rows of 2×2 determinants are interchanged at the cost of a sign. Now:

$$M = \begin{bmatrix} 0 & 1 - \begin{vmatrix} f_y^3 & f_z^3 \\ f_y^4 & f_z^4 \end{vmatrix} & \begin{vmatrix} f_y^2 & f_z^2 \\ f_y^4 & f_z^4 \end{vmatrix} & -\begin{vmatrix} f_y^2 & f_z^2 \\ f_y^3 & f_z^3 \end{vmatrix} \\ -1 + \begin{vmatrix} f_y^3 & f_z^3 \\ f_y^4 & f_z^4 \end{vmatrix} & 0 & -\begin{vmatrix} f_y^1 & f_z^1 \\ f_y^4 & f_z^4 \end{vmatrix} & \begin{vmatrix} f_y^1 & f_z^1 \\ f_y^3 & f_z^3 \end{vmatrix} \\ -\begin{vmatrix} f_y^2 & f_z^2 \\ f_y^4 & f_z^4 \end{vmatrix} & \begin{vmatrix} f_y^1 & f_z^1 \\ f_y^4 & f_z^4 \end{vmatrix} & 0 & -\begin{vmatrix} f_y^1 & f_z^1 \\ f_y^2 & f_z^2 \end{vmatrix} - 1 \\ \begin{vmatrix} f_y^2 & f_z^2 \\ f_y^3 & f_z^3 \end{vmatrix} & -\begin{vmatrix} f_y^1 & f_z^1 \\ f_y^3 & f_z^3 \end{vmatrix} & 1 + \begin{vmatrix} f_y^1 & f_z^1 \\ f_y^2 & f_z^2 \end{vmatrix} & 0 \end{bmatrix}$$

and

$$\vec{a} = \begin{bmatrix} f_x^1 \\ f_x^2 \\ f_x^3 \\ f_x^4 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} -f_y^3 - f_z^4 \\ f_z^3 - f_y^4 \\ f_y^1 - f_z^2 \\ f_y^2 + f_z^1 \end{bmatrix}.$$

Also, define:

$$\vec{b} = \begin{bmatrix} f_y^1 \\ f_y^2 \\ f_y^3 \\ f_y^4 \end{bmatrix}, \qquad \vec{c} = \begin{bmatrix} f_z^1 \\ f_z^2 \\ f_z^3 \\ f_z^4 \end{bmatrix}.$$

4.2 M as a Complex Structure

Define the quantities $h_{12}, h_{13}, h_{14}, h_{23}, h_{24}$, and h_{34} so that:

$$M = \begin{bmatrix} 0 & 1 - h_{12} & h_{13} & h_{14} \\ -(1 - h_{12}) & 0 & h_{23} & h_{24} \\ -h_{13} & -h_{23} & 0 & -(1 - h_{34}) \\ -h_{14} & -h_{24} & 1 - h_{34} & 0 \end{bmatrix}.$$

Now, we impose constraints by assuming symmetric or anti-symmetric correspondences between elements across the "counter-diagonal" of M. Specifically, assume:

$$\begin{aligned} h_{12} &= h_{34} \\ h_{13} &= h_{24} \\ h_{14} &= -h_{23}, \end{aligned}$$

so that M becomes:

$$M = \begin{bmatrix} 0 & 1 - h_{12} & h_{13} & h_{14} \\ -(1 - h_{12}) & 0 & -h_{14} & h_{13} \\ -h_{13} & h_{14} & 0 & -(1 - h_{12}) \\ -h_{14} & -h_{13} & 1 - h_{12} & 0 \end{bmatrix}. \tag{4.1}$$

Direct computation shows that M is a complex structure if and only if $(1 - h_{12})^2 + h_{13}^2 + h_{14}^2 = 1$.

Now, define the matrices R_i, R_j, and R_k as:

$$R_i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

$$R_j = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$R_k = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Straightforward computation shows that $R_i^2 = R_j^2 = R_k^2 = -I_{4\times 4}$, and $R_iR_j = R_k, R_jR_k = R_i$, and $R_kR_i = R_j$. In particular, R_i, R_j, and R_k are matrix representations of **i**, **j**, and **k** in the quaternion algebra. Now, (2.1) becomes:

$$\left[-(1-h_{12})R_i + h_{13}R_j - h_{14}R_k\right]\vec{a} = \vec{v}.$$

Recall that the inverse of a quaternion $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ is given by:

$$(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})^{-1} = \frac{a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}}{a^2 + b^2 + c^2 + d^2}.$$

This provides us with a means of inverting M to solve for $\vec{a}$:

$$\vec{a} = \left[-(1-h_{12})R_i + h_{13}R_j - h_{14}R_k\right]^{-1}\vec{v} = \frac{(1-h_{12})R_i - h_{13}R_j + h_{14}R_k}{(1-h_{12})^2 + h_{13}^2 + h_{14}^2}\vec{v}.$$

It will follow from Lemma 4.1 that the denominator never vanishes.

We now have an expression for the x-partial derivatives of f^1, f^2, f^3, and f^4 in terms of their other derivatives.

4.3 An Example

A first example will demonstrate the potential of this method to yield fruitful results. Define $u = y + z$, and suppose $f^1, \ldots, f^4$ are functions of only x and u:

$$f_i(x, y, z) = g_i(x, u),$$

for $i = 1, 2, 3, 4$. We have enough information to compute h_{12}, h_{13}, and h_{14}. To make the calculations easier, we shall first write out f's partial derivatives in terms of g's:

$$\begin{aligned} b_i &= \frac{\partial f_i}{\partial y} = \frac{\partial g_i}{\partial u}\frac{\partial u}{\partial y} = \frac{\partial g_i}{\partial u} \\ c_i &= \frac{\partial f_i}{\partial z} = \frac{\partial g_i}{\partial u}\frac{\partial u}{\partial z} = \frac{\partial g_i}{\partial u} \end{aligned}$$

That is, $\vec{b} = \vec{c}$. In particular, $\vec{b}$ and $\vec{c}$ are linearly dependent, so all of the h_{ij}'s are zero. In this case, the expression for M^{-1} is particularly simple, and we have:

$$\vec{a} = R_i \vec{v},$$

or

$$\begin{bmatrix} f_x^1 \\ f_x^2 \\ f_x^3 \\ f_x^4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -f_y^3 - f_z^4 \\ f_z^3 - f_y^4 \\ f_y^1 - f_z^2 \\ f_y^2 + f_z^1 \end{bmatrix} = \begin{bmatrix} f_y^4 - f_z^3 \\ -f_y^3 - f_z^4 \\ f_y^2 + f_z^1 \\ f_z^2 - f_y^1 \end{bmatrix}.$$

We now write out these four equations, in terms of $g_1, \ldots, g_4$:

$$\frac{\partial g_1}{\partial x} = \frac{\partial g_4}{\partial u} - \frac{\partial g_3}{\partial u} \tag{4.2}$$

$$\frac{\partial g_2}{\partial x} = -\frac{\partial g_4}{\partial u} - \frac{\partial g_3}{\partial u} \tag{4.3}$$

$$\frac{\partial g_3}{\partial x} = \frac{\partial g_2}{\partial u} + \frac{\partial g_1}{\partial u} \tag{4.4}$$

$$\frac{\partial g_4}{\partial x} = \frac{\partial g_2}{\partial u} - \frac{\partial g_1}{\partial u}. \tag{4.5}$$

It would be ideal to have explicit expressions for both partial derivatives of g_1 and g_2 in terms of derivatives of g_3 and g_4. To accomplish this, we simply consider (4.4) minus (4.5) and (4.4) plus (4.5):

$$\begin{aligned} 2\frac{\partial g_1}{\partial u} &= \frac{\partial g_3}{\partial x} - \frac{\partial g_4}{\partial x} \\ 2\frac{\partial g_2}{\partial u} &= \frac{\partial g_3}{\partial x} + \frac{\partial g_4}{\partial x}. \end{aligned}$$

Now, integrate (4.2) with respect to x to determine g_1 up to an arbitrary function $h_1(u)$:

$$g_1(x, u) = \int \left(\frac{\partial g_4}{\partial u} - \frac{\partial g_3}{\partial u} \right) dx + h_1(u).$$

To determine $h_1(u)$, differentiate the above and set it equal to our known expression for $\frac{\partial g_1}{\partial u}$:

$$\int \left(\frac{\partial^2 g_4}{\partial u^2} - \frac{\partial^2 g_3}{\partial u^2} \right) dx + h_1'(u) = \frac{1}{2} \left(\frac{\partial g_3}{\partial x} - \frac{\partial g_4}{\partial x} \right),$$

or

$$h_1'(u) = \frac{1}{2} \left(\frac{\partial g_3}{\partial x} - \frac{\partial g_4}{\partial x} \right) - \int \left(\frac{\partial^2 g_4}{\partial u^2} - \frac{\partial^2 g_3}{\partial u^2} \right) dx.$$

To ensure that h_1 is a function of u alone, we must set the x-partial derivative of the left-hand-side to zero:

$$0 = \frac{1}{2} \left(\frac{\partial^2 g_3}{\partial x^2} - \frac{\partial^2 g_4}{\partial x^2} \right) - \frac{\partial^2 g_4}{\partial u^2} + \frac{\partial^2 g_3}{\partial u^2} \tag{4.6}$$

We will return to this in a moment. For now, integrate (4.3) with respect to x:

$$g_2(x,u) = \int \left(-\frac{\partial g_4}{\partial u} - \frac{\partial g_3}{\partial u} \right) dx + h_2(u)$$

Compute the u-partial derivative and it set equal to our known expression for $\frac{\partial g_2}{\partial u}$:

$$\int \left(-\frac{\partial^2 g_4}{\partial u^2} - \frac{\partial^2 g_3}{\partial u^2} \right) dx + h_2'(u) = \frac{1}{2} \left(\frac{\partial g_3}{\partial x} + \frac{\partial g_4}{\partial x} \right),$$

or

$$h_2'(u) = \frac{1}{2} \left(\frac{\partial g_3}{\partial x} + \frac{\partial g_4}{\partial x} \right) - \int \left(-\frac{\partial^2 g_4}{\partial u^2} - \frac{\partial^2 g_3}{\partial u^2} \right) dx.$$

To ensure that $h_2'(u)$ does not depend on x, set the x-derivative of the right hand side to zero:

$$0 = \frac{1}{2} \left(\frac{\partial^2 g_3}{\partial x^2} + \frac{\partial^2 g_4}{\partial x^2} \right) + \frac{\partial^2 g_4}{\partial u^2} + \frac{\partial^2 g_3}{\partial u^2} \tag{4.7}$$

The sum and difference of (4.6) and (4.7) respectively yield:

$$\begin{aligned} 0 &= \frac{\partial^2 g_3}{\partial x^2} + 2\frac{\partial^2 g_3}{\partial u^2}, \\ 0 &= \frac{\partial^2 g_4}{\partial x^2} + 2\frac{\partial^2 g_4}{\partial u^2}, \end{aligned}$$

which are both transformed to Laplace's equation by the change of variables $v = \frac{1}{\sqrt{2}}u$. Thus, specifying any two harmonic functions $g_3(x,v)$ and

$g_4(x,v)$ determines $g_1(x,u)$ and $g_2(x,u)$. In other words, we are free to pick two arbitrary harmonic functions in this manner, and arrive at a solution to $Df = \sigma f$.

As an example, consider $g_3(x,v) = e^x \sin v$ and $g_4(x,v) = e^x \cos v$, where $v = \frac{1}{\sqrt{2}}u$.[1] Explicit computation shows:

$$
\begin{aligned}
\frac{\partial g_3}{\partial u} &= \frac{1}{\sqrt{2}} e^x \cos\left(\frac{u}{\sqrt{2}}\right), & \frac{\partial g_3}{\partial x} &= e^x \sin\left(\frac{u}{\sqrt{2}}\right)\\
\frac{\partial g_4}{\partial u} &= -\frac{1}{\sqrt{2}} e^x \sin\left(\frac{u}{\sqrt{2}}\right), & \frac{\partial g_4}{\partial x} &= e^x \cos\left(\frac{u}{\sqrt{2}}\right)\\
\frac{\partial^2 g_3}{\partial u^2} &= -\frac{1}{2} e^x \sin\left(\frac{u}{\sqrt{2}}\right), & \frac{\partial^2 g_4}{\partial u^2} &= -\frac{1}{2} e^x \cos\left(\frac{u}{\sqrt{2}}\right).
\end{aligned}
$$

It is readily computed that $h_1'(u) = h_2'(u) = 0$, in which case h_1 and h_2 are constants. Since these manifest as additive constants in f^1 and f^2, we shall ignore them. Now,

$$
\begin{aligned}
g_1(x,u) &= -\frac{1}{\sqrt{2}} \int \left(e^x \sin\left(\frac{u}{\sqrt{2}}\right) + e^x \cos\left(\frac{u}{\sqrt{2}}\right)\right) dx\\
&= -\frac{1}{\sqrt{2}} \left(e^x \sin\left(\frac{u}{\sqrt{2}}\right) + e^x \cos\left(\frac{u}{\sqrt{2}}\right)\right)\\
g_2(x,u) &= \frac{1}{\sqrt{2}} \int \left(e^x \sin\left(\frac{u}{\sqrt{2}}\right) - e^x \cos\left(\frac{u}{\sqrt{2}}\right)\right) dx\\
&= \frac{1}{\sqrt{2}} \left(e^x \sin\left(\frac{u}{\sqrt{2}}\right) - e^x \cos\left(\frac{u}{\sqrt{2}}\right)\right).
\end{aligned}
$$

Since $u = x + y$, the following is a solution to $Df = \sigma f$:

$$
\begin{aligned}
f(x,y,z) &= \frac{1}{\sqrt{2}} e^x \left(-\sin\left(\frac{y+z}{\sqrt{2}}\right) - \cos\left(\frac{y+z}{\sqrt{2}}\right)\right)\\
&+ \frac{1}{\sqrt{2}} e^x \left(\sin\left(\frac{y+z}{\sqrt{2}}\right) - \cos\left(\frac{y+z}{\sqrt{2}}\right)\right) \mathbf{i}\\
&+ e^x \sin\left(\frac{y+z}{\sqrt{2}}\right) \mathbf{j}\\
&+ e^x \cos\left(\frac{y+z}{\sqrt{2}}\right) \mathbf{k}.
\end{aligned}
$$

[1]Note: it is not necessary that g_3 and g_4 be the real and imaginary parts of an analytic function, as they are in the example.

4.4 Some Results on h_{12}, h_{13}, and h_{14}

A natural question to ask is: what exactly were Df and σf in the previous example? The answer lies in the following theorem:

Theorem 4.1. *Suppose f is a solution to (2.1), such that M has the complex structure defined by (4.1). Then $h_{12} = h_{13} = h_{14} = 0$ if and only if $Df = \sigma f = 0$.*

Proof. Recall that $Df = \sigma f$ can be rewritten in matrix from $M\vec{a} = \vec{v}$. Assuming $h_{12} = h_{13} = h_{14} = 0$, we can solve for $\vec{a}$ as $\vec{a} = M^{-1}\vec{v} = R_i\vec{v}$. Using the definition of R_i, we can write out the four components of the solution:

$$\begin{aligned} -f_x^2 &= f_y^3 + f_z^4 \\ f_x^1 &= -f_z^3 + f_y^4 \\ f_x^4 &= -f_y^1 + f_z^2 \\ -f_x^3 &= -f_z^1 - f_y^2. \end{aligned}$$

These equations are equivalent to, respectively, $\operatorname{tr} A = 0$, $\operatorname{tr} B = 0$, $\operatorname{tr} C = 0$, and $\operatorname{tr} D = 0$. Then $Df = 0$, so $Df = \sigma f = 0$.

On the other hand, if $Df = \sigma f = 0$, then the traces of A, B, C, and D are zero. Then the equation $\vec{a} = R_i\vec{v}$ holds. Also, recall:

$$\vec{a} = \frac{(1 - h_{12})R_i + h_{13}R_j - h_{14}R_k}{(1 - h_{12})^2 + h_{13}^2 + h_{14}^2}\vec{v}.$$

By the linear independence of R_i, R_j, and R_k, it follows that $h_{12} = h_{13} = h_{14} = 0$. □

Thus, in the previous example, Df and σf are both identically zero. Another way to see this is that $\vec{b} = \vec{c}$, while $\sigma f = \vec{a} \times \vec{b} \times \vec{c}$, which vanishes for linearly dependent $\vec{a}, \vec{b}, \vec{c}$. Since there is a constant linear dependence between $\vec{b}$ and $\vec{c}$, the example is included in the class of solutions constructed in Matthew Holden's thesis [5]. However, our example was found using a different approach. The following result states that any solution of the form $g(x, u)$ for u a linear combination of y and z will not be new.

Theorem 4.2. *If $f(x, y, z) = g(x, u)$, where u is a linear combination of y and z, is a solution to (2.1), then $\vec{b}$ and $\vec{c}$ differ by a real scalar (and are thus included in Holden's class of solutions). In particular, $Df = \sigma f = 0$.*

Proof. Suppose $u = my + nz$, where m and n are nonzero real constants, and that $f(x,y,z) = g(x,u)$ is a solution to (2.1). Then:

$$b_i = \frac{\partial f_i}{\partial y} = \frac{\partial g_i}{\partial u}\frac{\partial u}{\partial y} = m\frac{\partial g_i}{\partial u},$$

and

$$c_i = \frac{\partial f_i}{\partial z} = \frac{\partial g_i}{\partial u}\frac{\partial u}{\partial z} = n\frac{\partial g_i}{\partial u}.$$

Since $n \neq 0$, we have $\vec{b} = \frac{m}{n}\vec{c}$. In particular, since $\sigma f = \vec{a} \times \vec{b} \times \vec{c}$, we have $\sigma f = Df = 0$. □

To find solutions for which $Df = \sigma f \neq 0$, it must be that at least one of h_{12}, h_{13}, and h_{14} is not zero. The following results restrict the number of cases to be considered.

Lemma 4.1. *If any two of h_{12}, h_{13}, and h_{14} are zero, then the third is zero as well.*

Proof. We will demonstrate only one case, since the other two are identical in form. Suppose $h_{12} = h_{13} = 0$. Then the following determinants are zero:

$$\begin{vmatrix} f^3_y & f^3_z \\ f^4_y & f^4_z \end{vmatrix} = \begin{vmatrix} f^2_y & f^2_z \\ f^4_y & f^4_z \end{vmatrix} = 0.$$

It follows that the columns are linearly dependent: $f^3_y = kf^3_z, f^4_y = kf^4_z$, and $f^2_y = lf^2_z, f^4_y = lf^4_z$. Then $k = l$, so that $f^2_y = kf^2_z$ and $f^3_y = kf^3_z$. Then:

$$\begin{vmatrix} f^2_y & f^2_z \\ f^3_y & f^3_z \end{vmatrix} = \begin{vmatrix} kf^2_z & f^2_z \\ kf^3_z & f^3_z \end{vmatrix} = 0,$$

so $h_{14} = 0$. □

Theorem 4.3. *Suppose M has the complex structure (4.1). If $h_{12} = 0$, then $h_{13} = h_{14} = 0$.*

Proof. Suppose $h_{12} = 0$. Then also $h_{34} = 0$. Then from the definitions of h_{12} and h_{34}, we have:

$$\begin{aligned} f^1_y &= kf^1_z, \qquad f^3_y = lf^3_z, \\ f^2_y &= kf^2_z, \qquad f^4_y = lf^4_z. \end{aligned}$$

If $k = l$, it immediately follows that h_{13} and h_{14} are zero. Thus, suppose $k \neq l$. Now $h_{13} = h_{24}$ becomes:

$$\begin{aligned}
\begin{vmatrix} f_y^2 & f_z^2 \\ f_y^4 & f_z^4 \end{vmatrix} &= \begin{vmatrix} f_y^1 & f_z^1 \\ f_y^3 & f_z^3 \end{vmatrix}, \qquad \text{or} \\
\begin{vmatrix} kf_z^2 & f_z^2 \\ lf_z^4 & f_z^4 \end{vmatrix} &= \begin{vmatrix} kf_z^1 & f_z^1 \\ lf_z^3 & f_z^3 \end{vmatrix} \\
(k-l)f_z^2 f_z^4 &= (k-l)f_z^1 f_z^3 \\
f_z^2 f_z^4 &= f_z^1 f_z^3 \qquad (4.8)
\end{aligned}$$

Similarly, $h_{14} = -h_{23}$ becomes:

$$\begin{aligned}
-\begin{vmatrix} f_y^2 & f_z^2 \\ f_y^3 & f_z^3 \end{vmatrix} &= \begin{vmatrix} f_y^1 & f_z^1 \\ f_y^4 & f_z^4 \end{vmatrix} \\
-\begin{vmatrix} kf_z^2 & f_z^2 \\ lf_z^3 & f_z^3 \end{vmatrix} &= \begin{vmatrix} kf_z^1 & f_z^1 \\ lf_z^4 & f_z^4 \end{vmatrix} \\
-(k-l)f_z^2 f_z^3 &= (k-l)f_z^1 f_z^4 \\
-f_z^2 f_z^3 &= f_z^1 f_z^4 \qquad (4.9)
\end{aligned}$$

Multiply (4.8) by f_z^1 and use (4.9) to substitute for $f_z^1 f_z^4$:

$$-(f_z^2)^2 f_z^3 = (f_z^1)^2 f_z^3.$$

Then $f_z^3 = 0$ or $f_z^1 = f_z^2 = 0$. In the latter case, f_y^1 and f_y^2 are zero as well, so $h_{13} = h_{14} = 0$. So we are left with the case of $f_z^3 = 0$. By (4.8), it follows that f_z^2 or f_z^4 is zero. Then either h_{13} or h_{14} is zero. By the previous lemma, since two of the three h_{ij}'s are zero, the third is zero as well. □

From the same reasoning, it follows that if any of h_{12}, h_{13}, or h_{14} is zero, then the other two are zero as well. Thus, to consider solutions for which $Df = \sigma f \neq 0$ such that M has a complex structure, none of h_{12}, h_{13}, or h_{14} may be identically zero.

The proof of the last theorem gives insight into the $Df = \sigma f = 0$ case. It can be partitioned into cases where $k = l$ and where $k \neq l$. In what follows, we separately consider:

1) $k = l$, where k and l are smooth functions, and

2) $k \neq l$, where k and l are real constants.

4.5 $Df = \sigma f = 0$, where $k = l$

Suppose f's derivatives satisfy $\vec{b} = \varphi(x,y,z)\vec{c}$, where φ is a smooth, non-constant function (in the notation of the previous section, $k = l = \phi(x,y,z)$). Then $\sigma f = \vec{a} \times \vec{b} \times \vec{c} = 0$. Then in matrix form, (2.1) can be inverted to $\vec{a} = R_i \vec{v}$, or

$$f_x^1 = \varphi f_z^4 - f_z^3 \tag{4.10}$$
$$f_x^2 = -\varphi f_z^3 - f_z^4 \tag{4.11}$$
$$f_x^3 = \varphi f_z^2 + f_z^1 \tag{4.12}$$
$$f_x^4 = f_z^2 - \varphi f_z^1 \tag{4.13}$$

These four equations are actually $\operatorname{tr} B = 0, \operatorname{tr} A = 0, \operatorname{tr} D = 0$, and $\operatorname{tr} C = 0$. First, we will write out all of the first derivatives of f^1, in terms of derivatives of f^3 and f^4 only:

$$f_x^1 = \varphi f_z^4 - f_z^3$$
$$f_y^1 = \varphi f_z^1$$
$$f_z^1 = \frac{f_x^3 - \varphi f_x^4}{1+\varphi^2}.$$

Next, make the additional assumption that $f_x^4 = -\varphi f_x^3$, so that:

$$f_z^1 = \frac{f_x^3 + \varphi^2 f_x^3}{1+\varphi^2} = f_x^3.$$

Now, integrate (i) with respect to x to find f^1, up to some function $\alpha(y,z)$:

$$\begin{aligned} f^1(x,y,z) &= \int \left(\varphi f_z^4 - f_z^3\right) dx + \alpha(y,z) \\ &= \int \left(\varphi \frac{\partial f^4}{\partial z} - \frac{\partial f^3}{\partial z}\right) dx + \alpha(y,z). \end{aligned}$$

The next step is to ensure that $\frac{\partial f^1}{\partial y}$ and $\frac{\partial f^1}{\partial z}$ agree with our expressions for f_y^1 and f_z^1:

$$\begin{aligned} f_y^1 &= \varphi f_z^1 = \varphi f_x^3 = \varphi \frac{\partial f^3}{\partial x} = \int \left(\frac{\partial \varphi}{\partial y}\frac{\partial f^4}{\partial z} + \varphi \frac{\partial^2 f^4}{\partial y \partial z} - \frac{\partial^2 f^3}{\partial y \partial z}\right) dx + \frac{\partial \alpha}{\partial y}, \\ f_z^1 &= f_x^3 = \frac{\partial f^3}{\partial x} = \int \left(\frac{\partial \varphi}{\partial z}\frac{\partial f^4}{\partial z} + \varphi \frac{\partial^2 f^4}{\partial z^2} - \frac{\partial^2 f^3}{\partial z^2}\right) dx + \frac{\partial \alpha}{\partial z}. \end{aligned}$$

These can be arranged to expressions for $\frac{\partial \alpha}{\partial y}$ and $\frac{\partial \alpha}{\partial z}$, in terms of φ and the derivatives of f^3 and f^4. In principle, one can integrate and find such an α. However, there are a few necessary conditions. First, $\frac{\partial \alpha}{\partial y}$ and $\frac{\partial \alpha}{\partial z}$ must have no x-dependence, since α is a function of y and z only. We will set $\frac{\partial}{\partial x}$ of each of these to zero. Next, the derivatives of α must satisfy a certain relationship to ensure the existence of such an alpha. It can be shown that $\frac{\partial^2 \alpha}{\partial y \partial z} = \frac{\partial^2 \alpha}{\partial z \partial y}$ is a necessary and sufficient condition. First:

$$0 = \frac{\partial}{\partial x}\frac{\partial \alpha}{\partial y} = \frac{\partial \varphi}{\partial x}\frac{\partial f^3}{\partial x} + \varphi \frac{\partial^2 f^3}{\partial x^2} + \frac{\partial^2 f^3}{\partial y \partial z} - \frac{\partial \varphi}{\partial y}\frac{\partial f^4}{\partial z} - \varphi \frac{\partial^2 f^4}{\partial y \partial z} \tag{4.14}$$

$$0 = \frac{\partial}{\partial x}\frac{\partial \alpha}{\partial z} = \frac{\partial^2 f^3}{\partial x^2} + \frac{\partial^2 f^3}{\partial z^2} - \frac{\partial \varphi}{\partial z}\frac{\partial f^4}{\partial z} - \varphi \frac{\partial^2 f^4}{\partial z^2} \tag{4.15}$$

Now, on the one hand, we have:

$$\begin{aligned} \frac{\partial^2 \alpha}{\partial z \partial y} &= \frac{\partial \varphi}{\partial z}\frac{\partial f^3}{\partial x} + \varphi \frac{\partial^2 f^3}{\partial z \partial x} \\ &\quad - \int \left(\frac{\partial^2 \varphi}{\partial z \partial y}\frac{\partial f^4}{\partial z} + \frac{\partial \varphi}{\partial y}\frac{\partial^2 f^4}{\partial z^2} + \frac{\partial \varphi}{\partial z}\frac{\partial^2 f^4}{\partial y \partial z} + \varphi \frac{\partial^3 f^4}{\partial z^2 \partial y} - \frac{\partial^3 f^3}{\partial z^2 \partial y} \right) dx. \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{\partial^2 \alpha}{\partial y \partial z} &= \frac{\partial^2 f^3}{\partial y \partial x} \\ &\quad - \int \left(\frac{\partial^2 \varphi}{\partial y \partial z}\frac{\partial f^4}{\partial z} + \frac{\partial \varphi}{\partial z}\frac{\partial^2 f^4}{\partial y \partial z} + \frac{\partial \varphi}{\partial y}\frac{\partial^2 f^4}{\partial z^2} + \varphi \frac{\partial^3 f^4}{\partial y \partial z} - \frac{\partial^3 f^3}{\partial y \partial z^2} \right) dx. \end{aligned}$$

By rearranging the mixed partial derivatives of f^3 and f^4, we see that the integrals in these two expressions are equal. Then the condition $\frac{\partial^2 \alpha}{\partial y \partial z} = \frac{\partial^2 \alpha}{\partial z \partial y}$ takes the simpler form:

$$\frac{\partial \varphi}{\partial z}\frac{\partial f^3}{\partial x} + \varphi \frac{\partial^2 f^3}{\partial x \partial z} = \frac{\partial^2 f^3}{\partial x \partial y}. \tag{4.16}$$

To summarize, if we can find f^3, f^4, and φ satisfying (4.14), (4.15), (4.16), and $f^4_x = -\varphi f^3_x$, then $\alpha_1(y,z)$ exists, so f^1 exists. What about f^2? Our assumption of $f^4_x = -\varphi f^3_x$ led to $f^3_x = f^1_z$. Then (4.12) implies that f^2_z, and thus f^2_y are identically zero. In other words, $f^2 = f^2(x)$, and it can be found by integrating (4.11):

$$f^2(x) = \int \left(-\varphi f^3_z - f^4_z \right) dx.$$

We must ensure that the integrand does not depend on y or z. Setting the y and z derivatives of $-\varphi f_z^3 - f_z^4$ to zero will be two additional conditions:

$$\begin{aligned} 0 &= -\frac{\partial\varphi}{\partial y} f_z^3 - \varphi\frac{\partial f_z^3}{\partial y} - \frac{\partial f_z^4}{\partial y}, \qquad \text{or} \\ 0 &= \frac{\partial\varphi}{\partial y}\frac{\partial f^3}{\partial z} + \varphi\frac{\partial^2 f^3}{\partial y \partial z} + \frac{\partial^2 f^4}{\partial y \partial z} \qquad (4.17) \\ 0 &= -\frac{\partial\varphi}{\partial z} f_z^3 - \varphi\frac{\partial f_z^3}{\partial z} - \frac{\partial f_z^4}{\partial z}, \qquad \text{or} \\ 0 &= \frac{\partial\varphi}{\partial z}\frac{\partial f^3}{\partial z} + \varphi\frac{\partial^2 f^3}{\partial z^2} + \frac{\partial^2 f^4}{\partial z^2}. \qquad (4.18) \end{aligned}$$

At this point, we consider special cases that simplify (4.14), (4.15), or (4.16).

4.5.1 Case: $f_y^3 = 0$

Assume that $f_y^3 = 0$, so that f_z^3 is zero as well. Then $f^3 = f^3(x)$. It can be shown that solutions to $Df = \sigma f = 0$ have harmonic components, so $f^{3\prime\prime}(x) = 0$. Condition (4.16) now becomes:

$$\frac{\partial\varphi}{\partial z} f^{3\prime}(x) = 0.$$

We want to avoid $f_x^3 = 0$, so we assume $\frac{\partial\varphi}{\partial z} = 0$. Then $\varphi = \varphi(x,y)$. Equation (4.15) is now:

$$-\varphi\frac{\partial^2 f^4}{\partial z^2} = 0,$$

so we assume $\frac{\partial^2 f^4}{\partial z^2} = 0$. By the condition $f_x^4 = -\varphi f_x^3$, or $\frac{\partial f^4}{\partial x} = -\varphi(x,y) f^{3\prime}(x)$. We can integrate this to get an expression for f^4:

$$f^4(x,y,z) = -\int \varphi(x,y) f^{3\prime}(x) dx + r(y,z),$$

for some smooth function $r(y,z)$. Now, (4.14) reduces to:

$$0 = \frac{\partial\varphi}{\partial x} f^{3\prime}(x) - \frac{\partial\varphi}{\partial y}\frac{\partial r}{\partial z} - \varphi\frac{\partial^2 r}{\partial y \partial z}.$$

To tackle this equation, recall that (4.15) implies $\frac{\partial^2 f^4}{\partial z^2} = 0$, which in turn means that:

$$r(y,z) = m(y)z + n(y)$$

for some functions $m(y)$ and $n(y)$. From this expression, $\frac{\partial r}{\partial z} = m(y)$ and $\frac{\partial^2 r}{\partial y \partial z} = m'(y)$. Thus, equation (4.14) is now:

$$
\begin{aligned}
0 &= \frac{\partial \varphi}{\partial x} f^{3\prime}(x) - \frac{\partial \varphi}{\partial y} m(y) - \varphi m'(y) \\
&= \frac{\partial \varphi}{\partial x} f^{3\prime}(x) - \frac{\partial}{\partial y} \left(\varphi m(y) \right).
\end{aligned}
$$

The next step is to return our attention to (4.17) and (4.18). Since $f^3 = f^3(x)$, (4.17) reduces to

$$\frac{\partial^2 f^4}{\partial y \partial z} = 0,$$

which implies that $m'(y) = 0$. Then $m(y)$ identically equals some real constant m_0. Also, observe that (4.18) is automatically satisfied since $f^3 = f^3(x)$ and $\frac{\partial^2 f^4}{\partial z^2} = 0$.

Now, solve (a) for $f'(x)$:

$$f'(x) = \frac{m_0 \frac{\partial \varphi}{\partial y}}{\frac{\partial \varphi}{\partial x}}.$$

For this expression to be valid, the y-derivative of the right hand side must vanish. Set it to zero:

$$0 = \frac{m_0 \frac{\partial^2 \varphi}{\partial y^2} \frac{\partial \varphi}{\partial x} - \frac{\partial \varphi}{\partial y} \frac{\partial^2 \varphi}{\partial x^2}}{\left(\frac{\partial \varphi}{\partial x} \right)^2}.$$

Define $\xi = m_0 x + y$ and observe that $\varphi(x, y) = g(m_0 x + y) = g(\xi)$ is a solution for any smooth function g. Assuming that φ takes this form, we have:

$$f^{3\prime}(x) = \frac{m_0 g'(\xi)}{m_0 g'(\xi)} = 1.$$

Thus, we have:

$$
\begin{aligned}
f^3(x) &= x + \gamma_0, \qquad \text{and also} \\
f^4(x, y, z) &= -\int g(m_0 x + y) dx + m_0 z + n(y),
\end{aligned}
$$

and f^2 is given by:

$$\begin{aligned} f^2 &= -\int \left(\varphi\frac{\partial f^3}{\partial z} + \frac{\partial f^4}{\partial z}\right) dx \\ &= -\int m_0 dx \\ &= -m_0 x + \beta_0. \end{aligned}$$

Finally, f^1 is

$$\int m_0 g(m_0 x + y)dx + \alpha(y,z).$$

The last step is to determine $\alpha(y,z)$, given its partial derivatives.

$$\begin{aligned} \frac{\partial\alpha}{\partial y} &= g(m_0 x + y) f^{3\prime}(x) - \int g'(m_0 x + y)\frac{\partial f^4}{\partial z} dx \\ &= g(m_0 x + y) m_0 - \int g'(m_0 x + y) m_0 dx \\ &= g(m_0 x + y) m_0 - g(m_0 x + y) \\ &= (m_0 - 1) g(m_0 x + y), \end{aligned}$$

so

$$\alpha(y,z) = (m_0 - 1)\int g(m_0 x + y)dy + \alpha_1(z),$$

for some function $\alpha_1(z)$. Now we must force the z-derivative of $\alpha(y,z)$ to agree with the expression we already have for $\frac{\partial\alpha}{\partial z}$:

$$\alpha_1'(z) = m_0 - \int 0 dx,$$

so $\alpha_1(z) = m_0 z + \alpha_0$. Recall $\xi = m_0 x + y$, so that:

$$\int g(\xi)d\xi = \int m_0 g(m_0 x + y)dx = \int g(m_0 x + y)dy.$$

Then:

$$\begin{aligned} f^1(x,y,z) &= \int g(\xi)d\xi + (m_0 - 1)\int g(\xi)d\xi + m_0 z + \alpha_0 \\ &= m_0 \int g(\xi)d\xi + m_0 z + \alpha_0. \end{aligned}$$

At this point, it seems that $f = f^1 + f^2\mathbf{i} + f^3\mathbf{j} + f^4\mathbf{k}$ would be a solution to $Df = \sigma f = 0$. After trying an example with $g(m_0 x + y) = e^{m_0 x + y}$, it became clear that we were missing one final condition: the components of f must be harmonic functions. It is readily seen that f^2 and f^3 are harmonic on $\mathbb{R}^3$. If $m_0 \neq 0$,[2] consider Δf^1:

$$\Delta f^1 = -m_0 g'(\xi) - \frac{1}{m_0} g'(\xi) + n''(y).$$

For Δf^1 to equal zero, it is necessary that $m_0^2 = -1$ or $g'(\xi) = 0$. Both cases are contradictions, since m_0 is real, and $g'(\xi) = 0$ implies φ is a constant. Thus, the $f_y^3 = 0$ case leads only to linear solutions, which in principle can be written out explicitly.

4.5.2 Case: $\varphi = \varphi(x)$

Another way to simplify conditions (4.14), (4.15) and (4.16) is to assume φ is a function of x alone. Then we have:

$$0 = \varphi'(x)\frac{\partial f^3}{\partial x} + \varphi\frac{\partial^2 f^3}{\partial x^2} + \frac{\partial^2 f^3}{\partial y \partial z} - \varphi\frac{\partial^2 f^4}{\partial y \partial z} \tag{4.19}$$

$$0 = \frac{\partial^2 f^3}{\partial x^2} + \frac{\partial^2 f^3}{\partial z^2} - \varphi\frac{\partial^2 f^4}{\partial z^2}, \tag{4.20}$$

and

$$\varphi(x)\frac{\partial^2 f^3}{\partial x \partial z} = \frac{\partial^2 f^3}{\partial x \partial y}$$

The latter is:

$$\varphi(x)\frac{\partial}{\partial z}\left(\frac{\partial f^3}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f^3}{\partial x}\right). \tag{4.21}$$

This is satisfied by $\frac{\partial f^3}{\partial x} = g(\xi)$, where $\xi = \varphi(x)y + z$, for an arbitrary smooth function $g(\xi)$. Also, since we assume $f_x^4 = -\varphi(x)f_x^3$, we have:

$$\frac{\partial f^4}{\partial x} = -\varphi(x)g(\varphi(x)y + z)$$

Now, we can integrate with respect to x to find f^3 and f^4 up to functions of y and z:

$$f^3(x,y,z) = \int g(\varphi(x)y + z)dx + \gamma(y,z), \qquad \text{and}$$

$$f^4(x,y,z) = -\int \varphi(x)g(\varphi(x)y + z)dx + \delta(y,z),$$

[2]The $m_0 = 0$ case reduces f to a linear solution.

for some functions $\gamma(y,z)$ and $\delta(y,z)$. Differentiating these expressions for f^3 and f^4 will allow us to substitute into (4.19) and (4.20).

$$\begin{aligned}\frac{\partial^2 f^3}{\partial x^2} &= y\varphi'(x)g(\xi)\\ \frac{\partial f^3}{\partial z} &= \int g'(\xi)dx + \frac{\partial\gamma}{\partial z}\\ \frac{\partial^2 f^3}{\partial y\partial z} &= \int \varphi(x)g''(\xi)dx + \frac{\partial^2\gamma}{\partial y\partial z}\\ \frac{\partial^2 f^3}{\partial z^2} &= \int g''(\xi)dx + \frac{\partial^2\gamma}{\partial z^2},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial f^4}{\partial z} &= -\int \varphi(x)g'(\xi)dx + \frac{\partial\delta}{\partial z}\\ \frac{\partial^2 f^4}{\partial y\partial z} &= -\int \varphi(x)^2g''(\xi)dx + \frac{\partial^2\delta}{\partial y\partial z}\\ \frac{\partial^2 f^4}{\partial z^2} &= -\int \varphi(x)g''(\xi)dx + \frac{\partial^2\delta}{\partial z^2}.\end{aligned}$$

Now, (4.19) and (4.20) become:

$$\begin{aligned}0 &= \varphi'(x)g(\xi) + y\varphi(x)\varphi'(x)g(\xi) + \int \varphi(x)g''(\xi)dx + \frac{\partial^2\gamma}{\partial y\partial z}\\ &\quad \varphi(x)\left(-\int \varphi(x)^2g''(\xi)dx + \frac{\partial^2\delta}{\partial y\partial z}\right)\\ 0 &= y\varphi'(x)g(\xi) + \int g''(\xi)dx + \frac{\partial^2\gamma}{\partial z^2}\\ &\quad -\varphi(x)\left(-\int \varphi(x)g''(\xi)dx + \frac{\partial^2\delta}{\partial z^2}\right)\end{aligned}$$

The task at hand is to find suitable $g(\xi), \varphi(x), \gamma(y,z)$, and $\delta(y,z)$ that satisfy these equations. For now, it will help to recall the following: we still have $f^2_y = f^2_z = 0$, so that $f^2 = f^2(x)$. Since f^2 is given by integrating (4.11), we must have $\frac{\partial}{\partial y}(\varphi f^3_z + f^4_z) = \frac{\partial}{\partial z}(\varphi f^3_z + f^4_z) = 0$. Writing these out yields:

$$0 = \varphi(x)\frac{\partial^2 f^3}{\partial y\partial z} + \frac{\partial^2 f^4}{\partial y\partial z}, \tag{4.22}$$

$$0 = \varphi(x)\frac{\partial^2 f^3}{\partial z^2} + \frac{\partial^2 f^4}{\partial z^2}. \tag{4.23}$$

Using what we know about f^3 and f^4, (4.23) becomes:

$$\varphi(x) \int g''(\xi) dx + \frac{\partial^2 \gamma}{\partial z^2}(y,z) = \int \varphi(x) g''(\xi) dx - \frac{\partial^2 \delta}{\partial z^2}(y,z).$$

Now, differentiate this equation with respect to x:

$$\varphi'(x) \int g''(\xi) dx + \varphi(x) g''(\xi) = \varphi(x) g''(\xi),$$

so that

$$\varphi'(x) \int g''(\xi) dx = 0.$$

Since we want to avoid the case of $\varphi(x)$ being a constant, we take $\int g''(\xi)dx = 0$. Then $g''(\xi) = 0$, so:

$$g(\xi) = K\xi + L,$$

for some real constants K and L. Now, let us handle condition (4.22), which is greatly simplified by the fact $g''(\xi) = 0$:

$$\varphi(x) \frac{\partial^2 \gamma}{\partial y \partial z}(y,z) + \frac{\partial^2 \delta}{\partial y \partial z}(y,z) = 0.$$

Since the second term has no x-dependence, take the mixed partials of γ and δ to be zero.

Now, (4.19) and (4.20) are much easier to work with:

$$\begin{aligned} 0 &= \varphi'(x) g(\xi) + y\varphi(x)\varphi'(x) g(\xi) \\ 0 &= y\varphi'(x) g(\xi) + \frac{\partial^2 \gamma}{\partial z^2} - \varphi(x) \frac{\partial^2 \delta}{\partial z^2} \end{aligned}$$

Dividing through by $\varphi'(x)g(\xi)$ in condition (4.19) gives:

$$0 = 1 + y\varphi(x) \tag{4.24}$$

However, this contradicts the fact that φ depends on x only. So this approach has reached a dead end.

4.6 $Df = \sigma f = 0$, where $k \neq l$

Suppose that we have a solution f such that:

$$\begin{aligned} f_y^1 &= kf_z^1, \qquad f_y^3 = lf_z^3, \\ f_y^2 &= kf_z^2, \qquad f_y^4 = lf_z^4, \end{aligned}$$

where k and l are nonzero real constants with $k \neq l$. Then $Df = \sigma f = 0$, by Theorem 4.1. Since $k \neq l$, this case does not fall within the scope of Holden's thesis. Inverting to get $\vec{a} = R_i \vec{v}$, we have:

$$\begin{pmatrix} f_x^1 \\ f_x^2 \\ f_x^3 \\ f_x^4 \end{pmatrix} = \begin{pmatrix} f_y^4 - f_z^3 \\ -f_y^3 - f_z^4 \\ f_y^2 + f_z^1 \\ f_z^2 - f_y^1 \end{pmatrix} = \begin{pmatrix} lf_z^4 - f_z^3 \\ -lf_z^3 - f_z^4 \\ kf_z^2 + f_z^1 \\ f_z^2 - kf_z^1 \end{pmatrix}.$$

Call these equations (i), (ii), (iii), and (iv), respectively. Let us now solve for f_z^1 and f_z^2, so that we have all of the derivatives of f^1 and f^2 in terms of derivatives of f^3 and f^4. Equation (4.12) minus k times (4.13) yields:

$$f_z^1 = \frac{f_x^3 - kf_x^4}{1+k^2}.$$

Equation (4.12) plus k times (4.13) gives:

$$f_z^2 = \frac{kf_x^3 + f_x^4}{1+k^2}.$$

Let $H = \frac{1}{1+k^2}$. Now, integrate f_x^1 and f_x^2 with respect to x (using (i) and (ii)) to determine f^1 and f^2 up to unknown functions of y and z:

$$\begin{aligned} f^1(x,y,z) &= \int \left(lf_z^4 - f_z^3 \right) dx + \alpha(y,z) \\ f^2(x,y,z) &= \int \left(-lf_z^3 - f_z^4 \right) dx + \beta(y,z). \end{aligned}$$

We must ensure that α and β are such that $f_y^1 = \frac{\partial f^1}{\partial y}$, $f_z^1 = \frac{\partial f^1}{\partial z}$, $f_y^2 = \frac{\partial f^2}{\partial y}$, and $f_z^2 = \frac{\partial f^2}{\partial z}$. We first work with f_y^1 and f_z^1:

$$\begin{aligned} f_y^1 &= kf_z^1 = kH(f_x^3 - kf_x^4) = \frac{\partial f^1}{\partial y} = \int \left(l\frac{\partial^2 f^4}{\partial y \partial z} - \frac{\partial^2 f^3}{\partial y \partial z} \right) dx + \frac{\partial \alpha}{\partial y} \\ f_z^1 &= H(f_x^3 - kf_x^4) = \frac{\partial f^1}{\partial z} = \int \left(l\frac{\partial^2 f^4}{\partial z^2} - \frac{\partial^2 f^3}{\partial z^2} \right) dx + \frac{\partial \alpha}{\partial z}. \end{aligned}$$

These can be immediately solved for $\frac{\partial \alpha}{\partial y}$ and $\frac{\partial \alpha}{\partial z}$ (which will be of interest shortly). Next, consider f_y^2 and f_z^2:

$$\begin{aligned} f_y^2 &= kf_z^2 = kH(kf_x^3 + f_x^4) = \frac{\partial f^2}{\partial y} = \int \left(-l\frac{\partial^2 f^3}{\partial y \partial z} - \frac{\partial^2 f^4}{\partial y \partial z} \right) dx + \frac{\partial \beta}{\partial y} \\ f_z^2 &= h(kf_x^3 + f_x^4) = \frac{\partial f^2}{\partial z} = \int \left(-l\frac{\partial^2 f^3}{\partial z^2} - \frac{\partial^2 f^4}{\partial z^2} \right) dx + \frac{\partial \beta}{\partial z}. \end{aligned}$$

Note we can easily solve for $\frac{\partial \beta}{\partial y}$ and $\frac{\partial \beta}{\partial z}$.

To ensure that α and β exist, it is necessary that their x and y derivatives do not depend on x. Setting the x derivatives of $\frac{\partial \alpha}{\partial y}$, $\frac{\partial \alpha}{\partial z}$, $\frac{\partial \beta}{\partial y}$, and $\frac{\partial \beta}{\partial z}$ to zero, we arrive at the four conditions:

$$kH\left(\frac{\partial^2 f^3}{\partial x^2} - k\frac{\partial^2 f^4}{\partial x^2}\right) = l\frac{\partial^2 f^4}{\partial y \partial z} - \frac{\partial^2 f^3}{\partial y \partial z} \tag{4.25}$$

$$H\left(\frac{\partial^2 f^3}{\partial x^2} - k\frac{\partial^2 f^4}{\partial x^2}\right) = l\frac{\partial^2 f^4}{\partial z^2} - \frac{\partial^2 f^3}{\partial z^2} \tag{4.26}$$

$$kH\left(k\frac{\partial^2 f^3}{\partial x^2} + \frac{\partial^2 f^4}{\partial x^2}\right) = -l\frac{\partial^2 f^3}{\partial y \partial z} - \frac{\partial^2 f^4}{\partial y \partial z} \tag{4.27}$$

$$H\left(k\frac{\partial^2 f^3}{\partial x^2} + \frac{\partial^2 f^4}{\partial x^2}\right) = -l\frac{\partial^2 f^3}{\partial z^2} - \frac{\partial^2 f^4}{\partial z^2}. \tag{4.28}$$

Next, to ensure that α and β exist, their mixed partial derivatives must be consistent (e.g., $\frac{\partial^2 \alpha}{\partial y \partial z} = \frac{\partial^2 \alpha}{\partial z \partial y}$). Applying these conditions to α and β, we get:

$$\frac{\partial^2 \alpha}{\partial z \partial y} = kH\left(\frac{\partial^2 f^3}{\partial z \partial x} - k\frac{\partial^2 f^4}{\partial z \partial x}\right) = H\left(\frac{\partial^2 f^3}{\partial y \partial x} - k\frac{\partial^2 f^4}{\partial y \partial x}\right) = \frac{\partial^2 \alpha}{\partial y \partial z} \tag{4.29}$$

$$\frac{\partial^2 \beta}{\partial z \partial y} = kH\left(k\frac{\partial^2 f^3}{\partial z \partial x} + \frac{\partial^2 f^4}{\partial z \partial x}\right) = H\left(k\frac{\partial^2 f^3}{\partial y \partial x} + \frac{\partial^2 f^4}{\partial y \partial x}\right) = \frac{\partial^2 \beta}{\partial y \partial z}. \tag{4.30}$$

(Note the integral terms cancelled.) Rearranging the mixed partial derivatives, and recalling that $f^3_y = lf^3_z$, $f^4_y = lf^4_z$, we have:

$$\begin{aligned} k\frac{\partial}{\partial x}\left(\frac{\partial f^3}{\partial z} - k\frac{\partial f^4}{\partial z}\right) &= l\frac{\partial}{\partial x}\left(\frac{\partial f^3}{\partial z} - k\frac{\partial f^4}{\partial y}\right) \\ k\frac{\partial}{\partial x}\left(k\frac{\partial f^3}{\partial z} + \frac{\partial f^4}{\partial z}\right) &= l\frac{\partial}{\partial x}\left(k\frac{\partial f^3}{\partial z} + \frac{\partial f^4}{\partial z}\right). \end{aligned}$$

At this point the $k \neq l$ assumption is essential: both sides of (4.31) and (4.31) must be zero, and we have:

$$\begin{aligned} \frac{\partial^2 f^3}{\partial x \partial z} - k\frac{\partial^2 f^4}{\partial x \partial z} &= 0 \\ k\frac{\partial^2 f^3}{\partial x \partial z} + \frac{\partial^2 f^4}{\partial x \partial z} &= 0. \end{aligned}$$

Consider the first equation plus k times the second, and $-k$ times the first plus the second:

$$\begin{aligned}(1+k^2)\frac{\partial^2 f^3}{\partial x \partial z} &= 0\\(1+k^2)\frac{\partial^2 f^4}{\partial x \partial z} &= 0,\end{aligned}$$

so $\frac{\partial^2 f^3}{\partial x \partial z} = \frac{\partial^2 f^4}{\partial x \partial z} = 0$. We may substitute this into (4.29) and (4.30), then apply the same trick with multiplying by $\pm k$ and adding. We find that $\frac{\partial^2 f^3}{\partial x \partial y} = \frac{\partial^2 f^4}{\partial x \partial y} = 0$. It follows that dependencies of f^3 and f^4 separate as:

$$\begin{aligned}f^3(x,y,z) &= m(x)+n(y,z)\\f^4(x,y,z) &= p(x)+q(y,z)\end{aligned}$$

A necessary condition on f^3 an f^4 is that they be harmonic. In this case, $\frac{\partial^2 f^3}{\partial x^2} = \frac{\partial^2 f^4}{\partial x^2} = 0$. Then (4.25) and (4.27) become:

$$\begin{aligned}0 &= -\frac{\partial^2 f^3}{\partial y \partial z} + l\frac{\partial^2 f^4}{\partial y \partial z}\\0 &= l\frac{\partial^2 f^3}{\partial y \partial z} + \frac{\partial^2 f^4}{\partial y \partial z}\end{aligned}$$

Considering l times the first equation plus the second, and $-l$ times the second plus the first, we see that $\frac{\partial^2 f^4}{\partial y \partial z} = \frac{\partial^2 f^3}{\partial y \partial z} = 0$. Then f^3 and f^4 each separate as a sum functions of single variables. By the requirement that f^3 and f^4 be harmonic, it follows that they are linear combinations of x, y, and z, up to additive constants. In particular, $f^3_x, f^3_y, f^3_z, f^4_x, f^4_y$, and f^4_z are all real constants. Returning to the expressions for α and β, we see that $\frac{\partial \alpha}{\partial y}, \frac{\partial \alpha}{\partial z}, \frac{\partial \beta}{\partial y}$, and $\frac{\partial \beta}{\partial z}$ are all constants. It then follows that f^1 and f^2 are linear combinations of x, y, and z, up to additive constants. Then f is at best a linear solution to $Df = \sigma f = 0$.

4.7 $h_{12} = h_{13} = h_{14} = 1$

A solution with $h_{12} = h_{13} = h_{14} = 1$ would have $Df = \sigma f \neq 0$, and thus may be interesting. Note that M differs from a complex structure only by a scaling factor.

We begin by setting $h_{12} = h_{13} = h_{14} = 1$, and similarly $h_{34} = h_{24} = -h_{23} = 1$. These are six independent restrictions on four functions (not to mention the additional four restrictions from (2.1)). I have had no luck with this case.

4.8 Non-constant h_{ij}'s

Suppose we wish to consider solutions such that the h_{ij}'s are not all constants. Ideally, the denominator of:

$$\vec{a} = \left(\frac{(1 - h_{12})R_i - h_{13}R_j + h_{14}R_k}{(1 - h_{12})^2 + h_{13}^2 + h_{14}^2} \right) \vec{v}$$

would equal 1. Perhaps the simplest case is $h_{12} = 1$, $h_{13} = \sin\theta$, and $h_{14} = \cos\theta$, for $\theta = \theta(x, y, z)$. However, I ran into serious difficulties when attempting to find functions f^1, f^2, f^3, and f^4 whose derivatives satisfied these conditions, let alone also satisfying $\vec{a} = \big((1 - h_{12})R_i - h_{13}R_j + h_{14}R_k\big)\,\vec{v}$.

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