Cesaro Limits of Analytically Perturbed Stochastic Matrices

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Overview

- Motivating example
- Definitions
- The problem
- Eigenvalue background
- Current work on eigenvalues

Motivating example

The peculiar case of Roland the hot dog street vendor

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$$r_{n+1}(1) = (0.5 + \varepsilon) r_n(1) + (0.5 - \varepsilon) r_n(2)$$
$$r_{n+1}(2) = (0.5 - 2\varepsilon) r_n(1) + (0.5 + 2\varepsilon) r_n(2)$$

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$$r_{n+1}(2) = (0.5-2\varepsilon)r_n(1) + (0.5+2\varepsilon)r_n(2)$$

or ...

$$\begin{bmatrix} r_{n+1}(1) \\ r_{n+1}(2) \end{bmatrix} = \begin{bmatrix} 0.5+\varepsilon & 0.5-\varepsilon \\ 0.5-2\varepsilon & 0.5+2\varepsilon \end{bmatrix} \begin{bmatrix} r_n(1) \\ r_n(2) \end{bmatrix}$$

Motivating example (cont.)

The long-term daily average that Roland earns starting at corner 1 is

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From the previous recursive relationship,

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \begin{bmatrix} r_k(1) \\ r_k(2) \end{bmatrix} = \lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \begin{bmatrix} 0.5+\varepsilon & 0.5-\varepsilon \\ 0.5-2\varepsilon & 0.5+2\varepsilon \end{bmatrix}^k \begin{bmatrix} r_0(1) \\ r_0(2) \end{bmatrix}$$

Motivating example (cont.)

$$
\begin{aligned}
P^* &= \lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \begin{bmatrix} 0.5+\varepsilon & 0.5-\varepsilon \\ 0.5-2\varepsilon & 0.5+2\varepsilon \end{bmatrix}^k \\
&= \frac{1}{1-3\varepsilon} \begin{bmatrix} 0.5-2\varepsilon & 0.5-\varepsilon \\ 0.5-2\varepsilon & 0.5-\varepsilon \end{bmatrix}
\end{aligned}
$$

Motivating example (cont.)

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$$= \frac{1}{1-3\varepsilon} \begin{bmatrix} 0.5-2\varepsilon & 0.5-\varepsilon \\ 0.5-2\varepsilon & 0.5-\varepsilon \end{bmatrix}$$

Roland’s long-term average daily earnings are thus

$$\frac{0.5-2\varepsilon}{1-3\varepsilon} \cdot 90 + \frac{0.5-\varepsilon}{1-3\varepsilon} \cdot 100 = 95 + \frac{5\varepsilon}{1-3\varepsilon}$$

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What would happen if we let $\varepsilon \downarrow 0$ and $N \to \infty$ simultaneously?

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Definition: An *analytic perturbation* of a matrix $T_0 \in M_n(\mathbb{C})$ is a power series

$$T(\varepsilon) = T_0 + A(\varepsilon) = T_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots$$

in which the "coefficients" A_1, A_2, ... are in $M_n(\mathbb{C})$ as well.

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We want $P(\varepsilon)$ to be stochastic for all sufficiently small positive ε.

Foundation for My Thesis

- In 2002, Filar, Krieger, and Syed characterized the hybrid Cesaro limit

$$\lim_{\varepsilon \downarrow 0} \frac{1}{N(\varepsilon)} \sum_{k=1}^{N(\varepsilon)} P^k(\varepsilon)$$

for an analytically perturbed stochastic matrix $P(\varepsilon) = P_0 + A(\varepsilon)$

- Subject to the restriction that P_0 have no eigenvalues λ satisfying $|\lambda| = 1$ except for $\lambda = 1$.

My Thesis

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- If or when the limit does exist, how will such eigenvalues affect the limit?
- How does the rate at which $N(\varepsilon) \to \infty$ affect the existence or value of the limit?

Perturbed eigenvalues

If $T(\varepsilon) = T_0 + A(\varepsilon)$ and λ is an eigenvalue of T_0, then $T(\varepsilon)$ has a collection of eigenvalues $\lambda_1(\varepsilon)$, $\lambda_2(\varepsilon)$, $\ldots$, $\lambda_s(\varepsilon)$ that converge to λ as $\varepsilon \to 0$.

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Each $\lambda_j(\varepsilon)$ has a *Puiseux series*

$$\lambda_j(\varepsilon) = \lambda + c_{1,j}\varepsilon^{1/p_j} + c_{2,j}\varepsilon^{2/p_j} + \cdots$$

for some positive integer p_j and complex numbers $c_{1,j}$, $c_{2,j}$, . . .

Results for stochastic matrices

For an analytically perturbed stochastic matrix $P(\varepsilon)$, if $\lambda(\varepsilon) \neq 1$ is a perturbed eigenvalue corresponding to $\lambda = 1$, then the first nonzero coefficient in its Puiseux series has negative real part.

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It would be nice to have a similar result for other eigenvalues on the unit circle.

More on eigenvalues

In 1951, Karpelevic characterized, for a given positive integer n, the set of all complex numbers that are eigenvalues of an $n \times n$ stochastic matrix.

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- All *n*th or less roots of unity
- Curvilinear arcs connecting consecutive roots of unity

Region for $n = 4$