Noncommutative Geometry

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Abstract

We develop noncommutative field theory, starting from a very basic background and explore recent and important results in classical noncommutative field theory. The background section is of interest because it presents mathematical and physical interpretations of differential geometry together in a coherent way, not seen in most of the literature. We present several interesting examples that resulted from recent research in the field.

Contents

Chapter 1

Introduction

Noncommutativity is a central notion in both mathematics and physics. There are many important mathematical structures, which don't commute: most abstractly, nonabelian groups, which have numerous applications. In quantum mechanics, noncommutative algebras are one of the most important features. If two Hermitian operators do not commute, then there is an uncertainty relation between their corresponding observables. In quantum mechanics, linear position and momentum along the same direction do not commute giving the famous Heisenberg uncertainty relation. In certain situations, it is possible to have linear momentum along different directions not commute.

Noncommutative geometry is a generalization from these examples of noncommutativity in quantum mechanics where we consider the possibility that two spatial dimensions, like x and y, do not commute ($x \cdot y \neq y \cdot x$). This implies an uncertainty relation between the two directions on the quantum level, resulting in so-called "fuzzy space-time" and the abandonment of the concept of things being located at points (Szabo, 2003; Sykora, 2004). While noncommutative geometry was first studied before 1947, only recently has it become an active area of research in physics (Szabo, 2003).

In the 1980's, noncommutative geometry was considered as a way of extending the standard model in a number of different ways (Douglas and Nekrasov, 2001; Szabo, 2003). In condensed matter, noncommutative geometry arises naturally. For example, noncommutative geometry describes electrons in a magnetic field at lowest energy level which is related to the quantum Hall effect (Sykora, 2004). Recent interest in noncommutative geometry is strongly motivated by the discovery that string theory leads to noncommutative geometry in certain limits. In matrix models of M-theory (the theory encapsulating all six string theories), for example, compactification leads to noncommutative tori; open strings in magnetic fields are described by a noncommutative geometry with the Moyal product; and when considering Dp-branes interacting, the coordinates associate with noncommuting matrices (Gracia-Bondía et al., 2002). Fuzzy space-time also seems a natural way of limiting the ability to measure at the Planck length (Douglas and Nekrasov, 2001).

Despite all of these recent motivations, noncommutative geometry remains a nontrivial subject approached from numerous angles and allowing for numerous generalizations from the simplest, most well understood cases. On a fundamental level, the interest in noncommutative geometry results from the fact that it is a nonlocal theory and the theory of quantum gravity seems to require nonlocality. One of the potential problems with noncommutative field theory is that it breaks Lorentz invariance, which at least on macroscopic scales is believed to be a fundamental symmetry of the universe (Sykora, 2004; Douglas and Nekrasov, 2001).

This paper develops and examines some of the results in classical noncommutative field theory. We do not go into quantum field theory or the resulting fuzziness of space-time. Furthermore, we will not discuss time-space noncommutativity at a sustained level, since it is problematic and less well understood.

We begin by introducing basic mathematical and physical background material necessary to discuss noncommutative field theory on any level in Chapter 2. This chapter may be of interest even to those familiar with the material since it attempts to present the mathematics and physics in a more unified way than is usually done.

Chapter 3 gives a motivational example of noncommutative geometry arising in string theory when considering N D0-branes. Chapter 4 finally introduces noncommutative field theory and gives an example. Chapter 5 introduces the Seiberg-Witten map, which can map a noncommutative field theory to a commutative field theory. In this chapter, we demonstrate that this map may come at the expense of introducing non-flat curvature corresponding to gravitation. This, then, is another connection between noncommutative geometry and gravity. Chapter 6 presents a generalization of the Seiberg-Witten map that is a symmetry of noncommutative field theories. The symmetry is interesting because it leads to a conserved current in the space of noncommutative planes.

Chapter 2

Background Material

This section introduces the mathematics and physics that underlie the theory and notation the rest of the paper uses. We begin by introducing tensor fields on Riemannian manifolds as the mathematical background that frames the following discussion of classical field theory and the basic concepts of general relativity in the physics background section.

2.1 Mathematics Background

The following discussion mostly develops the concept of a tensor field on a manifold, which is an integral part of general relativity and clarifies the notation describing classical field theory.

2.1.1 Tensors: An Algebraic Approach

We begin by introducing the concept of a tensor. One can think of tensors as generalizations of vectors and matrices to higher dimensionality; however, before we can rigorously define a tensor, it is necessary to introduce and review several key concepts. To begin, we will assume familiarity with basic linear algebra.

Recall that a *vector space over a field* F is a set V that is closed for the operations of vector addition and multiplication by scalars, defined over it. Elements of V are the vectors and elements of the field, F (typically the reals or complex numbers), are the scalars. Vector addition is a mapping from $V \times V$ onto V and the multiplication is a mapping from $F \times V$ onto V. We will denote vector addition with $+$ and scalar multiplication implicitly, as usual. In addition to being closed over V, the operations must satisfy the following properties for all $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V$ and $a, b \in F$ (Lay, 2000):

1. **Commutativity**: $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$.
2. **Associativity**: $(\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3 = \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3)$.
3. **Identity**: There exists a vector $\mathbf{0}$ such that $\mathbf{v}_1 + \mathbf{0} = \mathbf{0} + \mathbf{v}_1 = \mathbf{v}_1$.
4. **Inverse**: There exists a vector $-\mathbf{v}_1$ such that $\mathbf{v}_1 + -\mathbf{v}_1 = \mathbf{0}$.
5. **Distributivity**: $a(\mathbf{v}_1 + \mathbf{v}_2) = a\mathbf{v}_1 + a\mathbf{v}_2$.
6. **Distributivity**: $(a + b)\mathbf{v}_1 = a\mathbf{v}_1 + b\mathbf{v}_1$.
7. **Associativity**: $(ab)\mathbf{v}_1 = a(b\mathbf{v}_1)$.
8. **Identity**: There exists a scalar 1 such that $1\mathbf{v}_1 = \mathbf{v}_1 1 = \mathbf{v}_1$.

The most familiar examples of vector spaces are $\mathbb{R}^n$ or $\mathbb{C}$ (which is isomorphic to $\mathbb{R}^2$). Additionally, the field F will always be a vector space over itself. Often we talk about a vector space V omitting the field and/or explicit definitions of the operations (for this paper, one may assume that the field is the reals) if they are clear from the context. Furthermore, we will frequently refer to V as both the set of vectors and as the vector space (including the two operations). This language, however, should not lead to confusion in most cases. Vector spaces, as every physicist and mathematician knows, have many useful and interesting properties that are beyond the scope of this paper to consider; however, we will review a few ideas used later in the paper.

Recall that a set of vectors $\{\mathbf{v}_i\}$ is *linearly independent* if the set of all linear combinations $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots$ has only the trivial zero, $c_i \equiv 0$ for all i (Lay, 2000). Further, remember that a *basis* for V is a linearly independent set of vectors whose set of all linear combination is V. In fact, for any given vector, there is precisely one linear combination of the basis vectors that equals it. The number of vectors in the basis is the dimension of V (Lay, 2000).

Vector spaces being defined, we introduce a *linear function* from vector space V to vector space W, that is, a function $f : V \to W$ with the additional property that for all $\mathbf{v}_1, \mathbf{v}_2 \in V$ and for all scalars a, b (Bishop and Goldberg, 1980),

$$f(\mathbf{v}_1 + \mathbf{v}_2) = f(\mathbf{v}_1) + f(\mathbf{v}_2), \tag{2.1a}$$
$$f(a\mathbf{v}_1) = af(\mathbf{v}_1). \tag{2.1b}$$

If f is bijective then we say that f is an *isomorphism* and we say that the two vector spaces are *isomorphic*. If two vector spaces are isomorphic, then they have the same structure and can be thought of as the same vector space; that is, we can get from V to W by relabeling all of the elements of V.

As one might expect, there are many linear functions possible from any one vector space V to another W, the set of which we will denote by $L(V, W)$. Surprisingly enough, $L(V, W)$ is a vector space in of itself. For $f, g \in L(V, W)$ and for all $\mathbf{v} \in V$, we define addition as $(f + g)(\mathbf{v}) = f(\mathbf{v}) + g(\mathbf{v})$ and multiplication by a scalar a as $(af)(\mathbf{v}) = af(\mathbf{v})$, as is natural. It is fairly easy to show that $L(V, W)$ is a vector space using these definitions; all of the properties derive directly from the fact that V and W are vector spaces and from f and g's linearity. The dimension of $L(V, W)$ is simply the product of the dimensions of V and W, assuming both are finite-dimensional (Dummit and Foote, 1999).

We can now define the concept of the *dual vector space* of V, written $V^* = L(V, F)$, as the set of all linear functions from the set V to vector space V's field, F (Bishop and Goldberg, 1980). Note the dimension of V is the same as the dimension of V^*, if the dimension of V is finite, since the dimension of F as a vector space over F as a field is 1. Only finite dimensional vector spaces are of interest here, so this paper will assume our vector spaces are finite-dimensional.

We call the dual of the dual vector space the double dual, written V^{**}. The double dual is naturally isomorphic (the isomorphism is natural because it can be defined without specifying a basis) to the original vector space. This means that we can think of elements of V as linear functions from V^* to F, while at the same time elements of V^* are linear functions from V to F. To see how this works, consider $\mathbf{v} \in V$ and $\bar{\mathbf{w}} \in V^*$. Now, a specific $\bar{\mathbf{w}}$ is a linear function from V to F, and so $\bar{\mathbf{w}}(\mathbf{v}) \in F$ for all $\mathbf{v} \in V$. Alternatively, we can fix $\mathbf{v}$, and let $\bar{\mathbf{w}}$ run over all of V^*, then we have a linear function(al) $g_{\mathbf{v}}(\bar{\mathbf{w}}) = \bar{\mathbf{w}}(\mathbf{v}) \in F$ for all $\bar{\mathbf{w}} \in V^*$. Notice that $g_{\mathbf{v}}$ is not an actual element of V; this is what we mean when we say that V is isomorphic with V^{**}, that to get from V to V^{**} we can simply replace $\mathbf{v}$ with $g_{\mathbf{v}}$, and vice versa.

At this point, we may finally introduce the concept of a *tensor* as a scalar-valued multilinear function with variables from vector space V and its dual V^*. By *multilinear*, we simply mean that a function f is linear in each of its variables,

$$f(\mathbf{v}_1, \mathbf{v}_2, \ldots, a\mathbf{v}_i + b\mathbf{w}_i, \ldots) = af(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_i, \ldots) + bf(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{w}_i, \ldots) \tag{2.2}$$

for all i from 1 to the number of variables of f (Bishop and Goldberg, 1980). The number of variables from V^* is called the *contravariant degree* and the number of variables from V is called the *covariant degree*. For example, a tensor on $V^* \times V \times V \times V$ has contravariant degree 1 and covariant degree 3, or, in short, is said to be of type $(1, 3)$. A tensor of type $(0, 0)$ is defined to be a scalar.

2.1.2 Tensors: A Physics Approach

The definition of a tensor given above is very general, which we would like to restrict to the case of interest to the physicist in this section. Before we proceed, we introduce manifolds and some associated objects.

For our purposes, a *manifold* is a space which is locally isomorphic to $\mathbb{R}^n$ and has smooth, well-defined mappings between coordinate patches that cover the manifold completely. For a more sophisticated definition and treatment of manifolds, see Carroll (1997), Bishop and Goldberg (1980), Eguchi et al. (1980), or a differential geometry text. The important idea behind a manifold is that it is not embedded in any other space (like $\mathbb{R}^n$) and allows for the most general geometry that we can do physics on. We allow an arbitrary number of coordinate patches because frequently just one coordinate system will not work over the whole manifold: eg. the unit circle, S^1. If we try to describe a circle with just one coordinate system, then there will be a point that is not well-defined or is not smooth. S^1 is a one-dimensional manifold since it is locally isomorphic to $\mathbb{R}$ and so position in the manifold may be described by one coordinate.

If we take a point p on a manifold M, we can imagine a curve on the manifold passing through p. At p, let us suppose that the coordinates of the coordinate patch containing p are x^μ, where the μ is a coordinate label and not an exponent. For example, x^0 is typically time, x^1 the first spatial dimension of the manifold (not that there is any particular ordering), and so on. The use of this notation is clarified below. Our curve, returning to the point at hand, can be described by $x^\mu = x^\mu(\lambda)$, where λ parameterizes the position along the curve. Without loss of generality, we assume that $x^\mu(0) = p$. The curve, then, defines a directional derivative on the space of smooth scalar (we only consider real manifolds for the rest of this paper) functions on M. More specifically, for a smooth function $f : M \to \mathbb{R}$, the directional derivative is

$$\frac{\partial f(x^\mu(\lambda))}{\partial \lambda}\Big|_{\lambda=0}.$$

We can now define the *tangent space* of M at p, $T_p(M)$, as the space of all directional derivatives arising from possible curves passing through p. The tangent space is a vector space (from the linearity of the derivative) of dimension equal to that of the manifold; its dual T_p^* is called the *cotangent space* (Carroll, 1997).

In physics, the tensors of interest are those whose vector space (V in section 2.1.1) is $T_p(M)$, that is multilinear real functions on $T_p^* \times T_p^* \times T_p \times T_p$, for example. Notice that because for each point on the manifold there is a new vector space associated with it, we also have a different tensor defined for each point of M. We call this a *tensor field*, in exactly the same way we talk about a vector field, where we assign a vector to each point in a space. To avoid possible confusion, note that frequently, we say "tensor" when we really mean a tensor field.

In order to work effectively with these tensor fields, it is necessary to specify a basis for T_p and T_p^*. While an orthonormal basis is possible and may seem convenient, this would mean that the basis would be differently defined for every point on the manifold since each point has a different vector space V. In physical applications, we almost exclusively use the directional derivative along the coordinate axes, $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$, as the basis for the tangent space. For the cotangent space, we use $\mathrm{d}x^\mu$ as a basis. Carroll (1997) makes a point of distinguishing $\mathrm{d}x^\mu$, a totally anti-symmetric covariant tensor formed by taking the exterior derivative of x^μ, from dx^μ, the familiar unrigorous notion of an infinitesimal displacement; however, for our purposes it suffices to think of it as simply the infinitesimal (Carroll, 1997; Eguchi et al., 1980). Although it may be somewhat intuitive, the reader will have to seek sources already mentioned for proof that these are elements of T_p and T_p^*, and are, in fact, bases. Furthermore, these bases obey the dual basis relationship

$$\mathrm{d}x^\mu(\partial_\nu) = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu, \tag{2.3}$$

where δ^μ_ν is the Kronecker delta, which is 0 if $\mu \neq \nu$ and 1 if $\mu = \nu$ (Carroll, 1997; Bishop and Goldberg, 1980). These bases are useful because they derive directly from the coordinate system and make a change of coordinates very easy as shown below.

Now that we have most of the necessary concepts to work with tensors it is necessary to introduce the notation that is used (and has started to be used without explanation). We begin by demonstrating with tensors of degree one first, and then generalize to a tensor of type (k, l) later. We can write a tensor V of type $(1, 0)$ as

$$V = V^\mu \partial_\mu = V^0 \partial_0 + V^1 \partial_1 + V^2 \partial_2 + \cdots + V^n \partial_n, \tag{2.4}$$

and a tensor W of type $(0,1)$ as

$$W = W_\mu \mathrm{d}x^\mu = W_0 \mathrm{d}x^0 + W_1 \mathrm{d}x^1 + W_2 \mathrm{d}x^2 + \cdots + W_n \mathrm{d}x^n, \qquad (2.5)$$

where our manifold is n-dimensional and we have introduced Einstein summation notation—repeated indices in the same term are summed over. We follow the convention given in Carroll (1997) and most other physics papers, where Greek indices sum from 0 up to the last coordinate (includes time) and Latin indices sum from 1 (does not include time). Note that the contravariant vector has superscript indices and the covariant vector has subscript indices. Also note that a tensor of type $(1,0)$ is simply an element of T_p, which is dual to T_p^*, so that it takes an element of T_p^* and maps it to $\mathbb{R}$, and vice versa. Explicitly, this works as

$$VW = V^\mu W_\nu \mathrm{d}x^\nu(\partial_\mu) = V^\mu W_\nu \delta^\mu_\nu = V^\mu W_\mu, \qquad (2.6)$$

which is an element of the reals since the weights on the bases are necessarily scalars as well. This type of product is called a *contraction* (Bishop and Goldberg, 1980). Contractions are physically significant because they are manifestly invariant, that is they do not depend on the choice of coordinate system. Measurable physical quantities obviously should not change when we change coordinates or bases.

Because one coordinate system will not necessarily cover the entire manifold, it is necessary to see how tensors undergo coordinate transformation. So, suppose that there are two coordinates x^μ and $x^{\mu'}$ (note the prime goes on the index). Our contravariant vector V must be the same tensor in either coordinate system, but we are undergoing a change of basis from ∂_μ to $\partial_{\mu'}$, that is, by the chain rule of differentiation,

$$V^\mu \partial_\mu = V^{\mu'} \partial_{\mu'} = V^{\mu'} \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu, \qquad (2.7)$$

and similarly

$$W_\mu \mathrm{d}x^\mu = W_{\mu'} \mathrm{d}x^{\mu'} = W_{\mu'} \frac{\partial x^{\mu'}}{\partial x^\mu} \mathrm{d}x^\mu. \qquad (2.8)$$

We see, then, that when we change coordinate a contravariant vector transforms as

$$V^\mu = V^{\mu'} \frac{\partial x^\mu}{\partial x^{\mu'}} \text{ or } V^{\mu'} = V^\mu \frac{\partial x^{\mu'}}{\partial x^\mu}, \qquad (2.9)$$

and a covariant vector transforms as

$$W_\mu = W_{\mu'} \frac{\partial x^{\mu'}}{\partial x^\mu} \text{ or } W_{\mu'} = W_\mu \frac{\partial x^\mu}{\partial x^{\mu'}}. \qquad (2.10)$$

Frequently in the literature, these transformations are the starting point for an introduction to tensors, and the emphasis of the discussion is on manipulation of indices and notation (Lovelock and Rund, 1989). Because we always use the same bases, the basis vectors ∂_μ and $\mathrm{d}x^\mu$ are dropped and we only write the scalar weights on the bases, V^μ and W_μ. This is analogous to the notation sometimes used where the vector (in $\mathbb{R}^3$) $a\hat{x}+b\hat{y}+c\hat{z}$ is written as (a, b, c), dropping the basis vectors $\hat{x}, \hat{y}$, and $\hat{z}$. Using the transformation rules, we see why the contraction of tensors is manifestly invariant since

$$V^\mu W_\mu = V^{\mu'} \frac{\partial x^\mu}{\partial x^{\mu'}} W_{\mu'} \frac{\partial x^{\mu'}}{\partial x^\mu} = V^{\mu'} W_{\mu'}, \tag{2.11}$$

so that it does not matter whether we do the contraction in x^μ or $x^{\mu'}$. Notice that this is exactly analogous to taking the dot product of two vectors in $\mathbb{R}^3$, which gives an invariant, the cosine of the angle between the vectors.

We can now generalize our notation to a general tensor T of type (k, l), which is written as

$$T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}, \tag{2.12}$$

where the contravariant components are written superscripted and the covariant components are written subscripted and are kept in order for clarity (the subscripts are not directly below the superscripts). The tensor transforms under a change of coordinates as

$$T^{\mu'_1 \cdots \mu'_k}{}_{\nu'_1 \cdots \nu'_l} = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{x'_k}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu'_1}} \cdots \frac{\partial x^{\nu_l}}{\partial x^{\nu'_l}}, \tag{2.13}$$

where the contravariant components each transform as the contravariant vector did and covariant components each transform as the covariant vector did (Carroll, 1997; Bishop and Goldberg, 1980; Lovelock and Rund, 1989). Contractions fall out naturally from Einstein summation notation; however, it is important to note that Einstein summation is only used between a contravariant index and a covariant index (anything else is shorthand notation or erroneous).

Before we proceed to the physics background, We need to introduce the metric tensor. In general relativity, the metric tensor, denoted by $g_{\mu\nu}$, gives all of the physically significant quantities. A general manifold does not necessarily have a metric tensor on it; when we define a metric on a manifold, we say it is a *Riemannian manifolds*. We will only do physics on Riemannian manifolds, because the metric tensor allows physically useful concepts such as distance, angle, geodesics (shortest distance between points on the manifold), and the concepts of "past" and "future" (Carroll, 1997). In general the metric tensor is a symmetric $(0, 2)$ tensor that we take to have non-zero determinant; there are no further restrictions that are placed on $g_{\mu\nu}$ (Carroll, 1997). By symmetric, we mean that the tensor is the same after exchanging indices, i.e. $g_{\mu\nu} = g_{\nu\mu}$. Not all tensors are symmetric; some are antisymmetric or have no particular symmetry—this is one of the reasons that the ordering of indices matters. Note that when we take the determinant of a tensor of degree 2, we treat the tensor as the matrix

$$\begin{pmatrix} g_{00} & g_{01} & \cdots \\ g_{10} & g_{11} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Requiring that the determinant be non-zero means that the metric tensor has an inverse satisfying

$$g^{\mu\nu} g_{\nu\lambda} = \delta^{\mu}_{\lambda}. \tag{2.14}$$

The principal utility of the metric tensor is in its ability to raise and lower indices. For a more comprehensive investigation of the metric tensor and Riemannian geometry, see (Carroll, 1997; Lovelock and Rund, 1989; Bishop and Goldberg, 1980). Raising indices works by contracting indices as follows

$$g^{\mu\nu} t_{\sigma\mu} = t_{\sigma}{}^{\nu}, \tag{2.15}$$

and lowering works analogously. So, the metric allows us to exchange contravariant and covariant components. The ability to raise and lower indices is the second reason for the ordering and "extra" spaces when placing indices on a tensor. Raising and lowering of indices allows us to make sense of the shorthand where we write a contraction between two contravariant indices or two covariant indices, since we can write

$$s^{\mu\nu} t^{\sigma\mu} = g_{\tau\mu} s^{\tau\nu} t^{\sigma\mu}. \tag{2.16}$$

In general relativity, the metric determines the curvature of space-time, which responds to mass and energy. For most of the paper, however, we do not be work in gravitational fields. We work in flat $3 + 1$ dimensional space-time, with constant metric

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that some authors have η with opposite sign, but it does not change the physics. The minus sign for the time component makes η consistent with special relativity. We can see this by writing the contraction

$$\eta_{\mu\nu}x^{\mu}x^{\nu} = -t^2 + x^2 + y^2 + z^2, \tag{2.17}$$

where we have labeled the spatial and temporal indices in the traditional manner. At this point, we should mention the fact that this paper is using units such that the speed of light, c, and Planck's constant over 2π, $\hbar$, are both unity, $c = \hbar = 1$. Theorists use these units to more clearly elucidate the mathematical structure of physical equations; however, the units are pretty inconvenient for experimentalists (who wants to measure everything in meters?). Now, bearing in mind the units, equation (2.17) should be recognized as the space-time interval in special relativity, which is invariant for all inertial reference frames. Since we have a total contraction, it is manifestly invariant, and therefore, we are working in Minkowski space-time, the geometry described by special relativity (Carroll, 1997). If the diagonal were all ones, then we would be working in familiar Euclidean geometry.

2.2 Physics Background

This section introduces Lagrangian mechanics and the action principle to develop classical field theory. We further discuss some of the important features of classical field theory, including Noether's Theorem and gauge fields. Initially, this section seems divorced from the previous; however, the notation and concepts we introduce in Section 2.1 become useful. We then conclude with a brief discussion of general relativity.

2.2.1 Classical Gauge Field Theory

Principle of Stationary Action

Recall that the basic equation of classical mechanics is Newton's second law

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \tag{2.18}$$

where $\mathbf{F}$ is the force on a mass m with momentum $\mathbf{p}$, t being time. This equation may be reformulated by introducing the *action* functional

$$S = \int_{\mathbf{x_1},t_1}^{\mathbf{x_2},t_2} L dt = \int_{\mathbf{x_1},t_1}^{\mathbf{x_2},t_2} (T - U) dt. \tag{2.19}$$

Here S is called the action and L, the Lagrangian. T is the kinetic energy, U is the potential energy, and our integral ranges from an initial position and time to a final position and time. It turns out that, for conservative forces, solving Equation (2.18) for the path that a particle takes through space-time from event $(t_1, \mathbf{x}_1)$ to event $(t_2, \mathbf{x}_2)$, assuming that such a path exists, is equivalent to finding the path that extremizes (almost always minimizes) the action. This powerful mathematical reformulation of Newtonian mechanics is called *Lagrangian mechanics*.

The *calculus of variations*, originally developed by Euler, was developed to find functions (or paths) that minimize functionals. The basic concept, is to suppose that a extremizing path exists, express a general function as a sum of the extremizing path and an arbitrary variation that is zero at the endpoints, and finding a partial differential equation that the minimizing function must satisfy as a result. In this process, we neglect all orders of the variation and its derivatives higher than linear order. This means that the function found makes the functional *stationary* (locally) to linear order. For a thorough development of the calculus of variations, see Hand and Finch (1998) or Bolza (1904).

Applying the calculus of variations to the general action in Equation (2.19), we can confirm that the path that makes the action stationary is equivalent to Newton's Second Law. This reformulation of mechanics is called the *principle of stationary* (sometimes least) *action*, Hamilton's Principle, or the "action principle." If we apply the calculus of variations to a general action describing a particle, we find the Euler-Lagrange Equations (Hand and Finch, 1998)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} \equiv 0 \qquad (\text{for all } i). \tag{2.20}$$

where, q_i is the ith generalized coordinate. While the principle of stationary action is equivalent to Newton's second law, action formulations are preferable for theoretical purposes since they are more naturally quantized into a quantum mechanical theory (Sahakian, 2004; Kaku, 1993). Even in classical mechanics, Lagrangian mechanics is preferred for certain problems because we can easily use generalized coordinates that are more natural and there are only as many as there are degrees of freedom (Hand and Finch, 1998). For example, if we imagine trying to find the motion of a marble in a frictionless hemispherical bowl, we can use two generalized coordinates θ and ϕ, whereas we might use three in Newtonian mechanics. Before moving on, one should make a final note that while Lagrangian mechanics only describes conservative forces, all forces are conservative on the most

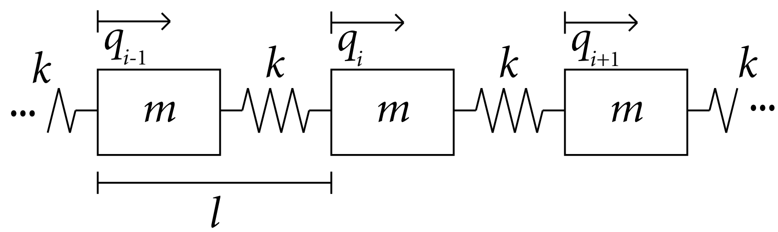

Figure 2.1: A diagram showing masses connected by springs in one dimension. Each mass m is the same and each spring has the same spring constant, k. When all the springs are at equilibrium, the masses are separated by a distance l and the coordinate of the ith mass q_i is 0, for all i.

fundamental level. These facts conspire to make action formulations the theoretical physicist's preferred tool.

The Principle of Least Action applied to Tensor Fields on Manifolds

Frequently, we wish to describe a physical tensor field and not a particle. Field theories are used in strong-electroweak unification, string theories, and theories describing phase transitions. Below, we see a succinct formulation of electrodynamics using field theory and Noether's theorem as reasons to be interested in field theories; however, before proceeding, we begin by motivating classical field theory with a more mechanical and intuitive example from continuum mechanics.

Consider a large number of particles each of mass m connected by identical springs, with spring constant k, in a line as shown in Figure 2.1. We restrict the motion of the particles to the one dimension along the line, so that only compression may occur. We denote the displacement of the ith particle's position from equilibrium by $q_i(t)$. Using the formulation of mechanics developed above, we describe the system's motion by specifying the Lagrangian

$$L = T - U = \sum_{i=1}^{N} \frac{m}{2}\dot{q}_k^2 - \sum_{i=1}^{N} \frac{k}{2}(q_{i+1} - q_i)^2, \tag{2.21}$$

where $\dot{q}_i = \frac{\partial q_i}{\partial t}$ and there are a total of N particles. We might use this system to model linear compression or stretching of a beam, but perhaps we feel that we could more accurately model the beam as a continuum rather than a large number of discrete particles. To accomplish this, we might take the limit as l goes to 0, while holding $\lambda = m/l$ and $Y = kl$ fixed. So as l gets smaller the mass of each particle decreases and the spring constant of each spring increases. Rewriting Equation (2.21), we formally see that as $l \rightarrow 0$ in this way

$$L \longrightarrow \int dx \left(\frac{\lambda}{2}\dot{q}^2 - \frac{Y}{2}q'^2 \right), \tag{2.22}$$

where λ is the mass density of our beam, Y is its Young's Modulus, and $q(x,t)$ is a new scalar field replacing the $q_i(t)$. We use q' to mean $\frac{\partial q(x,t)}{\partial x}$ and note that the extra l term became the differential line element dx. We now have a scalar field being used to describe the local distortion of the beam from equilibrium at position x along the beam's length at time t.

Now that we have a physical quantity described by a tensor field, it becomes necessary to find the equations of motion of the field, and so we apply our action principle for tensor fields as well as for particles. We mostly are interested in scalar fields or tensor fields of type $(0,0)$; however, the concept can be generalized. To this end, we introduce the Lagrangian density

$$L = \int \mathscr{L}(\varphi, \partial_\mu \varphi) d\tau, \tag{2.23}$$

where $\mathscr{L}$ is the Lagrangian density and the integral is over all space. This definition is specific to flat space-time; in Section 5, the definition is generalized to non-flat space-times. If our Lagrangian density is a function of several scalar fields φ^a (a enumerates different fields, not indexes the space-time coordinates), then applying the action principle and the calculus of variations gives us the fields' equations of motion, a reformulation of the Euler-Lagrange Equations,

$$\frac{\partial \mathscr{L}}{\partial \varphi^a} = \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi^a)} \right), \tag{2.24}$$

where μ is summed over by the Einstein Summation convention and a is not (the two a's are in separate terms). We now have an equation of motion for each field φ^a as a result (Sahakian, 2004; Kaku, 1993). Note that one of the consequences of moving to a noncommutative geometry is that there are no longer general equations of motion like Equation (2.24).

This framework of Lagrangian mechanics with tensor fields is called classical field theory and has been extremely successful at describing a plethora of physical phenomena as mentioned previously. A typical example of a classical field theory with two fields might be

$$\mathscr{L} = \eta^{\mu\nu}\partial_\mu\varphi^a\partial_\nu\varphi^a + \frac{m^2}{2}\varphi^a\varphi^a + V(\varphi^a). \tag{2.25}$$

The first term is called the kinetic term, which is required for dynamics. The coefficient of the first term $\eta^{\mu\nu}$ is our Lorentz metric, so we are working in Minkowski space. The second term is a mass term that gives the fields φ^a mass m. The last term is a potential term that would have higher powers of φ^a, to put the field φ in a potential energy distribution. Recall that using the Einstein summation notation, μ, ν, and a are summed over in the first term and a is summed over in the second term. If we are describing two massive fields with no potential energy then $a = 1, 2$ and $V = 0$ (Sahakian, 2004; Rubakov, 1999).

Noether's Theorem

Noether's theorem, proved by Emmy Noether in 1918, is one of the most important theorems in physics demonstrating the importance of symmetry. Verbally, Noether's Theorem states that for every continuous symmetry of a Lagrangian density $\mathscr{L}$, there exists a conserved current, called the Noether current (Hand and Finch, 1998; Kaku, 1993).

In order to understand this theorem mathematically, we need to define a transformation of our Lagrangian density. We write our transformation as $\{\delta x^\mu, \delta\varphi^a\}$, which corresponds to

$$x^{\mu\prime} = x^\mu + \delta x^\mu(x) \tag{2.26}$$

$$\varphi^{a\prime}(x) = \varphi^a(x) + \bar{\delta}\varphi^a = \varphi^a(x) + \delta\varphi^a - \delta x^\mu\partial_\mu\varphi^a, \tag{2.27}$$

where we replace the φ^a and x^μ with their primed versions in the action. Note that $\delta\varphi^a = \varphi^{a\prime}(x') - \varphi^a(x)$, while $\bar{\delta}\varphi^a = \varphi^{a\prime}(x) - \varphi^a(x)$. To get Equation (2.27), we Taylor expand $\varphi^{a\prime}(x') = \varphi^a(x + \delta x)$ about x and drop quadratic and higher terms since the transformation is continuous and therefore δx can be made arbitrarily small.

If the transformation is a symmetry of the action, then the equations of motion given in Equation (2.24) are unchanged by the transformation. If transformation $\{\delta x^\mu, \delta\varphi^a\}$ is a symmetry of the action, then the associated Noether current is given by

$$j^\mu = \frac{\partial\mathscr{L}}{\partial\partial_\mu\varphi^a}\bar{\delta}\varphi^a + \mathscr{L}\delta x^\mu. \tag{2.28}$$

To say that j^μ is conserved means that the divergence of j is zero, $\partial_\mu j^\mu = 0$. Noether's theorem is now fairly easily proved through manipulation of tensor indices; however, the original proof did not rely on this modern notation and was much more involved (Sahakian, 2004; Kaku, 1993; Lovelock and Rund, 1989).

As an example of applying Noether's Theorem, consider the Lagrangian density in Equation (2.25), two massive scalar fields with no potential energy, we can find a Noether current (actually, we can find several, but the example is a particularly interesting one). We begin by defining $\phi = \varphi^1 + i\varphi^2$, and rewriting the Lagrangian as

$$\mathscr{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi^\dagger\partial_\nu\phi + \frac{m^2}{2}\phi^\dagger\phi. \tag{2.29}$$

We may now see an interesting symmetry. Namely, $\phi' = e^{-i\alpha}\phi$ is a symmetry of the action. This transformation is interesting because the two fields are "mixed." The associated Noether current, then, is

$$j^\mu = \phi\partial_\mu\phi^\dagger - \phi^\dagger\partial_\mu\phi, \tag{2.30}$$

which exhibits a similar mixing. Another example of mixing occurs when considering electrodynamics with special relativity and the electric and magnetic fields become mixed during Lorentz transformations (switching inertial reference frames). Some other typical results easily derived from Noether's theorem are that translational symmetry implies conservation of linear momentum, rotational symmetry implies conservation of angular momentum, and temporal symmetry implies conservation of energy (Kaku, 1993).

Gauge Fields

Gauge theory is part of what makes classical field theory so effective. The standard model of particle physics describing electromagnetism, the weak force, and the strong force, is gauged by $SU(2)_L \otimes U(1) \otimes SU(3)_C$, where U is the unitary group and SU is the special unitary group. For more details about how this works, see Kaku (1993); Griffiths (1987). A gauge field is a field with respect to which a symmetry is implemented The gauge group is the symmetry group of the gauge field; in modern gauge theory physics, we require that all physically measurable quantities, the action, and the equations of motion are invariant under a gauge transformation (an element of the symmetry group) (Rubakov, 1999). Generally this added symmetry leads to the conservation of some charge through Noether's theorem (in electrodynamics, electric charge). As an example of a classical gauge field theory, we will consider the classic example of electromagnetism, a field theory gauged by $U(1)$, the unitary group (Kaku, 1993; Schüker, 2004).

We begin by letting A^μ be the potential field, containing both the scalar electric potential, V ($\mathbf{E} = \nabla V$), and the magnetic vector potential, $\mathbf{A}$ ($\mathbf{B} = \nabla \times \mathbf{A}$); and j^μ (no longer a generic Noether current) to contain both the electric charge density, ρ, and the electric current density, $\mathbf{j}$, as

$$A^\mu = (V, \mathbf{A}) \tag{2.31}$$

$$j^\mu = (\rho, \mathbf{j}). \tag{2.32}$$

We may write the conservation of charge simply as $\partial_\mu j^\mu = 0$ and Maxwell's equations become

$$\partial_\mu F^{\mu\nu} = j^\nu, \tag{2.33}$$

where we define $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (Griffiths, 1999). To see the relationship between this formulation of electromagnetism and the usual one, we should comment that if we write $F^{\mu\nu}$ as a matrix we find

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}, \tag{2.34}$$

where the electric field is $\mathbf{E} = (E^1, E^2, E^3)$ and the magnetic field is $\mathbf{B} = (B^1, B^2, B^3)$ (Sahakian, 2004; Kaku, 1993).

The Lagrangian density that gives Maxwell's equations in Equation (2.33) is

$$\mathscr{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu, \tag{2.35}$$

where $c = 1$, as mentioned above. Now, there is local symmetry of the gauge field, A_μ, that is, we can add the gradient of any function $\delta A_\mu = \partial_\mu \Lambda(x)$ to A_μ and the equations of motion will be unchanged. It is a *local symmetry* because our transformation may depend on position and time. If we examine the definition of $F^{\mu\nu}$, we can see that $F^{\mu\nu}$ will be invariant under this symmetry. For the second term, we use integration by parts to get a $\partial_\mu j^\mu$, which we know to be 0, and the fact that the remaining integral vanishes for Λ that decrease rapidly enough as $\sqrt{|x^\mu x_\mu|} \to \infty$. We now have a lot of gauge freedom since for any A^μ, there is a family of equivalent gauge fields differing by the gradient of an arbitrary function. These gauge transformations form a group isomorphic to $U(1)$ for any fixed point, where the group operation is addition (Rubakov, 1999). The group properties follow naturally:

- **Closure** $\partial_\mu \Lambda_1 + \partial_\mu \Lambda_2 = \partial_\mu(\Lambda_1 + \Lambda_2) = \partial_\mu \Lambda_3$.
- **Associativity** $(\partial_\mu \Lambda_1 + \partial_\mu \Lambda_2) + \partial_\mu \Lambda_3 = \partial_\mu \Lambda_1 + (\partial_\mu \Lambda_2 + \partial_\mu \Lambda_3)$.
- **Identity** $\partial_\mu \Lambda + \partial_\mu k = \partial_\mu \Lambda + 0 = \partial_\mu \Lambda$, where k is a constant.
- **Inverse** $\partial_\mu \Lambda + \partial_\mu(-\Lambda) = \partial_\mu \Lambda - \partial_\mu \Lambda = 0$.

To see that this group is isomorphic to $U(1)$ for a fixed x, recall that any element of $U(1)$ can be written as $e^{i\alpha}$ for some $\alpha \in \mathbb{R}$, since $U(1)$ is the group of all 1×1 unitary matrices. The group operation is multiplication. The isomorphism is the map $f : \partial_\mu \Lambda \mapsto e^{i\Lambda(x)}$. Some more work is required to make this map well-defined, but eventually one sees that the two groups are isomorphic (Rubakov, 1999). The $U(1)$ in the standard model gauge group comes from electrodynamics.

One of the consequences of having a local gauge group is that the kinetic term $\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$ may not be invariant under gauge transformations. For example, consider a complex scalar field Lagrangian density like Equation (2.29) interacting with A_μ, the electrodynamics gauge field

$$\mathscr{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi^\dagger\partial_\nu\phi + \frac{m^2}{2}\varphi^\dagger\varphi - \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - j^{\mu\prime}A_\mu, \tag{2.36}$$

where $j^{\mu\prime} = -i(\phi^\dagger\partial_\mu\phi - \partial_\mu\phi^\dagger\phi)$. If we allow local gauge transformations of the form $A_\mu{}' \mapsto A_\mu + \partial_\mu\alpha(x)$ simultaneously with $\phi' \mapsto e^{i\alpha(x)}\phi$ and $\phi^{\dagger\prime} \mapsto e^{-i\alpha(x)}\phi^\dagger$, then $\partial_\mu\phi' = e^{i\alpha}(\partial_\mu\alpha + \partial_\mu\phi)$ and then $\partial_\mu\phi$ is not invariant. The solution is to replace ∂_μ with a *covariant derivative* D_μ that is invariant for gauge transformations and reduces to ∂_μ when there is no gauge field. Typically, this means adding a constant term with the gauge field to ∂_μ. For this example, we define $D_\mu = \partial_\mu - A_\mu$ and then our new Lagrangian density will be gauge invariant. Note that the other two parts of the standard model, $SU(2)$ and $SU(3)$, are both nonabelian groups, which complicate the covariant derivative beyond this example (Rubakov, 1999).

2.2.2 General Relativity

In this subsection, we very briefly review general relativity. For those interested in a more thorough discussion, see Carroll (1997) or any introductory general relativity text.

In general relativity, we allow for non-flat metrics, as mentioned above. Roughly, this means that the metric depends on position on the manifold, leading to curvature. Einstein's equations relate the intrinsic curvature of the space-time manifold to the distribution of energy and matter on the manifold. Intrinsic curvature is very different from extrinsic curvature. For example, there is no intrinsic curvature of S^1, or any other one dimensional manifold (Carroll, 1997). Intrinsic curvature measures among other things how rapidly parallel geodesics converge. *Geodesics* are the shortest distance between two points on a manifold, or the path that light travels in general relativity, e.g.: straight lines in Euclidean space and great circles on S^2. In order to write Einstein's equations, then, we need to have some rigorous notion of intrinsic curvature.

Before we can develop intrinsic curvature, we must first add additional structure to the manifold, a "connection". We begin by consider the partial derivative operator ∂_μ acting on a general tensor. As it turns out, for tensors of degree greater than one, $T^{\mu\nu}$ for example, $\partial_\mu T^{\mu\nu}$ is not a tensor. This is revealed when checking the transformation laws for tensor components in Equation (2.13). It becomes necessary, then, to introduce another *covariant derivative*, ∇_μ of a general tensor T, which is still a tensor. Note the difference between this covariant derivative and the covariant derivative for gauge theory (in fact, they are related since coordinate transformations in general relativity correspond to gauge transformations). We require this covariant derivative, ∇_μ, to be linear and satisfy the Leibnitz product rule. As it turns out, then, our covariant derivative is of the form

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda, \tag{2.37}$$

or for a general tensor T

$$\begin{aligned}\nabla_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} &= \partial_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} \\ &+ \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1\lambda\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} + \cdots \\ &- \Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1\mu_2\cdots\mu_k}{}_{\lambda\nu_2\cdots\nu_l} - \Gamma^{\lambda}_{\sigma\nu_2} T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\lambda\cdots\nu_l} - \cdots ,\end{aligned} \tag{2.38}$$

where $\Gamma^\nu_{\mu\lambda}$ is a linear correction to the partial derivative, called the *connection coefficients* (Carroll, 1997). It is important to know that the connection coefficients are not tensors, hence the index placement. Now there are many possible covariant derivatives and many possible connections that can be defined on our manifold, but in general relativity, we are interested in the uniquely defined connection induced by the metric called the *Christoffel* connection. This is the connection that gives the covariant derivative some other nice properties, like being able to raise and lower indices without effecting the derivative. Carroll (1997) derives the Christoffel symbols and defines them as

$$\Gamma^{\nu}_{\mu\lambda} = \frac{g^{\nu\rho}}{2}\left(\partial_\mu g_{\lambda\rho} + \partial_\lambda g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right), \tag{2.39}$$

where we can clearly see that if we are in a flat space-time with metric $\eta_{\mu\nu}$, the covariant derivative reduces to the partial derivative. This suggests that the Christoffel connection is related to the intrinsic curvature. To proceed quickly, then, we define the *Riemann* or *curvature tensor* as

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}, \tag{2.40}$$

without motivation or derivation. From here we may define the *Ricci tensor*, as the contraction

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}, \tag{2.41}$$

and the *Ricci scalar*, as the further contraction

$$R = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu}, \tag{2.42}$$

noting that R is only disambiguated by the number of indices. Now, we may finally write Einstein's Equations, the equations of motion for the metric in general relativity,

$$G_{\mu\nu} = R_\mu \nu - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{2.43}$$

where $G_{\mu\nu}$ is called the *Einstein tensor*, G_N is Newton's gravitational constant, and $T_{\mu\nu}$ is the energy-momentum tensor. The energy-momentum tensor describes the energy and momentum on the manifold completely, in such a way that $\nabla_\mu T^{\mu}{}_{\nu} = 0$ is the conservation of energy and momentum in curved space-time (Carroll, 1997).

Before we leave general relativity, we should note that Equation (2.43) can be derived from a field theory, using the Hilbert Action S_H,

$$L_{\text{total}} = \frac{1}{8\pi G_N} L_H + L_M, \tag{2.44}$$

where $\mathscr{L}_H = \sqrt{-g} R$ (g denoting the determinant of the metric) and S_M will depend on the kind and distribution of matter and energy.

Chapter 3

Noncommutative Geometry in String Theory

While noncommutative geometry in physics has been around since before 1947, only recently has significant interest developed because of connections with string theory. In 1998, several authors demonstrated that certain situations in string theory directly engender noncommutative field theories (Douglas and Nekrasov, 2001). While there are many motivations for studying noncommutative geometry in the context of string theory, this chapter presents one example where noncommutative geometry arises naturally from string theory.

3.1 A Brief Introduction to String Theory

String theory is an evolving theory; however, the original premise of string theory is to replace particles in the standard model of physics with vibrational modes of one-dimensional strings in $9+1$ dimensions (nine spatial dimensions and one temporal dimension). There can be both closed strings and open strings with free ends. Closed strings have all of the properties of the graviton, which is why string theory is a candidate theory for quantum gravitation. Within this framework, there are six different types of string theory, which were found to be different perturbative expansions of one $10+1$ dimensional theory called "M theory" (Gauntlett, 1998).

Working in M theory proves difficult, so for our example we work in "Type IIA" string theory. This string theory has both open strings and Dp-branes. Dp-branes arise in string theory as a consequence of considering the boundary conditions of the ends of open strings. Dp-branes, in particular, are p-dimensional solitonic, extended solutions (strings are perturbative solutions) of string theory, which open strings' ends may have Dirichlet boundary conditions on (the "D" stands for Dirichlet). Two D-branes interact with each other when open strings connect them. The strings are directed and come in pairs of opposing directionality (Gauntlett, 1998; Zwiebach, 2004).

3.2 Noncommutative Geometry from String Theory: An Example

Our example is of N D0-branes interacting. We define $N \times N$ Hermitian matrices X^i corresponding to each of the nine spatial dimensions such that the jth diagonal element is the position of the D0-brane along the ith spatial direction. The off diagonal elements are interaction terms between two D0-branes. Writing X^i out, then,

$$X^i = \begin{pmatrix} x_1 & c_{12} & c_{13} & \cdots \\ c_{12}^* & x_2 & c_{23} & \cdots \\ c_{13}^* & c_{23}^* & x_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where x_j is the position of the jth D0-brane along the ith direction (not to be confused with x^i, the spatial coordinates) and c_{kl} is the interaction between the kth and the lth D0-branes. The complex conjugate is the interaction from the open string with the opposite orientation connecting them. Since these are arbitrary Hermitian matrices, they will not necessarily commute. Here, then, we have a noncommutative geometry since the coordinate directions have become matrices.

More explicitly, the Lagrangian density for N D0-branes interacting is (Myers, 1999)

$$\mathscr{L} = -T_0 \mathrm{Tr} \left(1 - \frac{\lambda^2}{2} \dot{X}^i \dot{X}^i - \frac{\lambda^2}{4} [X^i, X^j][X^i, X^j] + \mathcal{O}(\lambda^3) \right), \tag{3.1}$$

where Tr indicates the trace, $\dot{X}^i$ is the covariant derivative of X^i with respect to time, λ is a parameter related to fundamental constants in string theory, and T_0 is the tension in a D0-brane. We use the variational principle or the Euler-Lagrange equations to find that the equations of motion are

$$\ddot{X}^i = [X^j, [X^i, X^j]], \tag{3.2}$$

where we immediately can see the noncommutative nature of these nonlinear equations of motion.

An interesting static solution to Equation (3.2) occurs if all but two of the spatial dimensions commute and we take N to approach infinity. The solution, in this case, is given by the Heisenberg algebra

$$[X^i, X^j] = \theta^{ij} I_{N \times N}, \tag{3.3}$$

where if $i = j$ or either i or j is greater than 2 then $\theta^{ij} = 0$, otherwise θ^{ij} is constant and antisymmetric. $I_{N\times N}$ is the N by N identity matrix. This solution is an infinite noncommuting plane of D0-branes all interacting. An interesting result of considering this solution is that it appears to describe a D2-brane (Myers, 1999). This suggests the possibility that all D-branes can be built up from D0-branes, where the branes form a noncommutative geometry by the nature of their interactions. The noncommutative space resulting from the matrix description of Dp-branes can be reformulated through a noncommutative product on a space of functions on the commutative plane. In this case, the noncommutative space is called the *Moyal plane* which has an associated Moyal product, all of which is explored more in the next chapter.

Chapter 4

Noncommutative Field Theories

In this section, we take the classical field theory developed in Section 2.2 and put it on a noncommutative geometry. We then examine some of the effects.

4.1 Star Product

The simplest way to study field theories on noncommutative geometry is to introduce a noncommutative product and replace ordinary multiplication with it. Field theories of this kind will be called *noncommutative field theories*, which should not be confused with nonabelian field theories (nonabelian refers to the gauge group here). Although we want our product to be noncommutative, we still require some basic properties:

- Associativity is preserved: $(f \star g) \star h = f \star (g \star h)$.
- Bilinearity or distributivity: $af \star bg = abf \star g$, where $a, b \in \mathbb{C}$ and f, g are functions of x^μ.
- The Leibnitz product rule is preserved: $\partial_\mu(f \star g) = \partial_\mu f \star g + f \star \partial_\mu g$,

where $\star$ denotes the noncommutative "star product". Additionally, we want our product to be continuously noncommutative with some parameter θ, so that when $\theta = 0$ our star product reduces to ordinary commutative multiplication.

There are many possible candidate star products (infinitely many, in fact) that can be implemented. Gracia-Bondía et al. (2002) suggest that some of the more complicated star products and their associated noncommutative geometries should be explored; however, we are most interested in the simplest products motivated by three algebras from physics. Star products, in general, use an infinite number of derivatives to create noncommutativity as we see in the three most commonly used $\star$-products (Vacaru, 2000)

$$f\star g = \begin{cases} exp[\frac{i}{2}\frac{\partial}{\partial u^i}\theta^{ij}\frac{\partial}{\partial u'^j}]f(u)g(u')|_{u'\to u}, & \text{canonical structure;} \\ exp[\frac{i}{2}u^k g_k(i\frac{\partial}{\partial u'}, i\frac{\partial}{\partial u''})]f(u')g(u'')|_{u'\to u''}, & \text{Lie structure;} \\ q^{\frac{1}{2}(-u'\frac{\partial}{\partial u'}v\frac{\partial}{\partial v}+u\frac{\partial}{\partial u}v'\frac{\partial}{\partial v'})}f(u,v)g(u',v')|_{(u',v')\to(u,v)}, & \text{quantum structure.} \end{cases} \tag{4.1}$$

The first product, called the *Moyal product*, is the simplest and most commonly used in the literature, but notice that the general structure of the three star products is the same. Here the u^i are coordinates and θ^{ij} is the noncommutative parameter. Note that in order for this to make sense, the exponential must be expanded as a Taylor series, leading to an infinite number of derivatives. In fact, it is sufficient to specify the algebra or the commutation relations to determine the general star product. For example, the Moyal product corresponds with the Heisenberg algebra, $[x^i, x^j]_\star = i\theta^{ij}$, where $[A, B]_\star = A \star B - B \star A$ is the commutator. When Gracia-Bondía et al. (2002) suggest generalizations of the Moyal product, they consider star products of the form $[x^i, x^j]_\star = ic^{ij}_k x^k$, for constant c's for three or more spatial dimensions. Most treatments only use two noncommuting spatial dimensions, as will we, because it illuminates the effects of noncommutative geometry while adding as few complications as is possible. Making time a noncommutative variable will also not be considered since it leads to effects that are not of interest. For a very thorough development of different $\star$-products, see Sykora (2004). Additionally, we will focus on the Moyal product for the same reason and because the Seiberg-Witten Map is defined for the Moyal product (Seiberg and Witten, 1999). There is another way to generalize the $\star$-product: we can allow θ to depend on position $\theta(x)$ (Fosco and Torroba, 2004).

We can use the above examples of $\star$-products to illustrate some general properties of the noncommutative parameter:

- If $\theta^{\mu\nu} = 0$, $f \star g = fg$, exactly as required.
- If θ^{ij} is symmetric, $f \star g = g \star f$. We will consider only anti-symmetric θ with zero on the diagonal.
- $x^1 \star x^2 = x^1 x^2 + i\theta^{12}/2$, and therefore $[x^1, x^2]_\star = \frac{i}{2}(\theta^{12} - \theta^{21}) = i\theta$,

where in the last line, we introduce notation frequently used in the literature when there are only two noncommuting spatial dimensions (Pinzul and Stern, 2004).

4.2 Effects of Applying the Star Product to Classical Field Theories

At this point, we return to the classical field theory developed in section 2.2, but replace ordinary multiplication with noncommutative $\star$-products. This will have very significant consequences, the first is that we have to generalize the calculus of variations to allow for an infinite number of derivatives; standard treatments only consider functionals of up to one derivative. Furthermore, the Euler-Lagrange Equations will no longer hold because commutativity and only single derivatives are used in the derivation. The derivation of Noether's theorem similarly fails. Therefore, we must return to the action principle to make any progress.

For example, we can take our two field system in Equation (2.25) introduced in Section 2.2 and put it on a noncommutative geometry. Our new Lagrangian density then is

$$\mathscr{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi^a \star \partial_\nu\varphi^a + \frac{m^2}{2}\varphi^a \star \varphi^a + k\varphi^a \star \varphi^a \star \varphi^b \star \varphi^b, \tag{4.2}$$

where we have introduced a quartic interaction term. If we complexify the field as we did before, letting $\phi = \varphi^1 + i\varphi^2$ we already begin to see some of the nontrivial effects of noncommutative geometry. The Lagrangian then becomes

$$\begin{aligned}\mathscr{L} =& \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi^\dagger\partial_\nu\phi + \frac{1}{2}[\partial_\mu\phi, \partial_\nu\phi^\dagger]) + \frac{m^2}{2}(\phi^\dagger\phi + \frac{1}{2}[\phi, \phi^\dagger]) \\ &+ \frac{k}{4}((\phi\phi^\dagger)^2 + (\phi^\dagger\phi)^2 + \phi(\phi^\dagger)^2\phi + \phi^\dagger(\phi)^2\phi^\dagger),\end{aligned} \tag{4.3}$$

where we have suppressed the $\star$'s for compactness and ease of notation. Note that we still have the same symmetry $\phi' = e^{-i\alpha}\phi$; however, Noether's theorem no longer applies, so it is no longer clear what the conserved current is if there is one. In order to find the equations of motion, we must return to the action principle $\delta S = 0$ for a variation $\phi' = \phi + \delta\phi$ and $\phi^{\dagger\prime} = \phi^\dagger + \delta\phi^\dagger$. Assuming that we are using the Moyal product, then the effect of the star product is null for quadratic terms (including the commutator terms): we plug in with the definition of the Moyal star product, and then repeatedly integrate by parts, recalling that overall derivative terms do not effect the action.

Applying the variation, we find terms in the perturbed action of the form

$$\phi \star \phi^\dagger \star \delta\phi \star \phi^\dagger,$$

where the $\delta\phi$ or $\delta\phi^\dagger$ is not on the left or right side. In order to find the equations of motion, it is necessary to factor out $\delta\phi$ and $\delta\phi^\dagger$. If we use a specific definition of the star product, the resolution is to Fourier transform φ and vary the Fourier transformed action. At the end, one can inverse transform back to get the equations of motion.

Other more fundamental issues arise when considering noncommutative field theories. As a result of the infinite number of derivatives in the action, the equations of motion are non-local. By non-local, we mean that distant points affect each other directly. This is precisely the aspect of noncommutative field theories that make them good for studying quantum gravity, since quantum gravity seems to require non-locality. Additionally, gauge transformations and global translations in a noncommutative direction are equivalent, which is similar to general relativity (Rivelles, 2004). As we see in Chapter 5, there are other intriguing connections between noncommutative field theories and gravity that arise when considering gauge fields.

Chapter 5

The Seiberg-Witten Map

The Seiberg-Witten map is one of the most important transformations of noncommutative geometries. It is a map from a noncommutative theory with parameter θ to an equivalent field theory with a different noncommutative parameter $\bar{\theta}$. Since commutative gauge theories are fairly well understood, the Seiberg-Witten map from noncommutative theories to commutative theories with $\theta = 0$ is an important tool to understanding noncommutative geometry.

To begin, consider a noncommutative gauge theory with a scalar field φ coupled to the electrodynamics gauge field in $3+1$ dimensions. We can write the total action as the sum

$$S_{\text{total}} = S_A + S_\varphi = -\frac{1}{4}\int d^4x\, \hat{F}^{\mu\nu} \star \hat{F}_{\mu\nu} + \frac{1}{2}\int d^4x\, \hat{D}^\mu \hat{\varphi} \star \hat{D}_\mu \hat{\varphi}, \tag{5.1}$$

where we are using $\hat{}$ to indicate the original noncommutative fields. The field strength generalizes from the commutative field strength in Section 2.2 to $\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu]_\star$ and the covariant derivative becomes $\hat{D}_\mu \hat{\varphi} = \partial_\mu \hat{\varphi} - i[\hat{A}_\mu, \hat{\varphi}]_\star$, with gauge field $\hat{A}_\mu$. We let $\hat{\lambda}$ be the gauge parameter, that is, $\hat{\lambda}$ generates the gauge transformation. This means we write the gauge transformations as

$$\delta \hat{A}_\mu = \hat{D}_\mu \hat{\lambda}, \qquad \text{and} \qquad \delta \hat{\varphi} = -i[\hat{\varphi}, \hat{\lambda}]_\star. \tag{5.2}$$

Given this example, we write the differential equations for the general

Seiberg-Witten map without confusion,

$$\delta\hat{A}_i(\theta) = \delta\theta^{kl}\frac{\partial\hat{A}_i(\theta)}{\partial\theta^{kl}} = -\frac{1}{4}\delta\theta^{kl}\{\hat{A}_k, (\partial_l\hat{A}_i + \hat{F}_{li})\}_\star, \tag{5.3a}$$

$$\delta\hat{\lambda}(\theta) = \delta\theta^{kl}\frac{\partial\hat{\lambda}(\theta)}{\partial\theta^{kl}} = \frac{1}{4}\delta\theta^{kl}\{\partial_k\lambda, A_l\}_\star, \tag{5.3b}$$

$$\delta\hat{F}_{ij}(\theta) = \delta\theta^{kl}\frac{\partial\hat{F}_{ij}(\theta)}{\partial\theta^{kl}} = \frac{1}{4}\delta\theta^{kl}\left[2\{\hat{F}_{ik}, \hat{F}_{jl}\}_\star - \{\hat{A}_k, (\hat{D}_l\hat{F}_{ij} + \partial_l\hat{F}_{ij})\}_\star\right], \tag{5.3c}$$

where $\{A, B\}_\star = A \star B + B \star A$ is the $\star$-product anti-commutator and the variables without hats are the equivalent commutative field theory's variables (Seiberg and Witten, 1999; Rivelles, 2004). The second equalities can be solved in principle to give the map, assuming the map exists; there will be some theories which do not have a map to a commutative field theory. Note that the differential equation and the map are both highly nonlinear and so exact maps are rare. Seiberg and Witten (1999), however, show that for a gauge field of degree one with constant $\hat{F}$ the explicit map for Equation (5.3c) is

$$\hat{F} = \frac{1}{1 + F\theta}F, \tag{5.4}$$

where $F\theta$ is the matrix product of F and θ.

For the noncommutative electrodynamics action, for example,

$$S = -\frac{1}{4}\int d^4x\hat{F}^{\mu\nu} \star \hat{F}_{\mu\nu},$$

it is possible to use the exact Seiberg-Witten map to find that (Banerjee, 2004)

$$-\frac{1}{4}\int d^4x\,\hat{F}^{\mu\nu} \star \hat{F}_{\mu\nu} = \frac{1}{4}\int d^4x\,\sqrt{\det(1 + F\theta)}\left(\frac{1}{1 + F\theta}F\frac{1}{1 + F\theta}F\right); \tag{5.5}$$

however, this is a special case. Generally the Seiberg-Witten map will be expanded in θ.

Returning to the action in Equation (5.1), we write the Seiberg-Witten map to the equivalent commutative theory as an expansion in θ by applying the Moyal product definition of the star product (Rivelles, 2004; Seiberg and Witten, 1999; Rivelles, 2003):

$$\hat{A}_\mu = A_\mu - \frac{1}{2}\theta^{\alpha\beta}A_\alpha(\partial_\beta A_\mu + F_{\beta\mu}) + \mathcal{O}(\theta^2), \tag{5.6a}$$

$$\hat{\varphi} = \varphi - \theta^{\alpha\beta}A_\alpha\partial_\beta\varphi + \mathcal{O}(\theta^2). \tag{5.6b}$$

The Seiberg-Witten map in Equation (5.6) makes our commutative field theory have the conventional gauge transformation $\delta A_\mu = \partial_\mu \Lambda$ and $\delta\varphi = 0$. Applying Equation (5.6) to the action, we find the equivalent commutative theory has actions, to first order in θ,

$$S_\varphi = \frac{1}{2}\int d^4x \left[\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + 2\eta^{\tau\nu}\theta^{\mu\alpha}F_{\alpha\tau}\left(-\partial_\mu\varphi\partial_\nu\varphi + \frac{1}{4}\eta_{\mu\nu}\eta^{\sigma\rho}\partial_\sigma\varphi\partial_\rho\varphi\right)\right], \tag{5.7}$$

and

$$S_A = -\frac{1}{4}\int d^4x \left[F^{\mu\nu}F_{\mu\nu} + 2\theta^{\mu\rho}F_\rho{}^\nu\left(F_\mu{}^\sigma F_{\sigma\nu} + \frac{1}{4}\eta_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right)\right], \tag{5.8}$$

where indices are raised and lowered with the flat metric η and $\theta = 0$ collapses all of the $\star$'s to ordinary multiplication.

The action for a field theory on a curved manifold, with a gravitational background, is of the form

$$S_{g,\varphi} = \int d\tau\,\sqrt{-g}\mathscr{L}(g_{\mu\nu},\varphi,\partial_\mu\varphi),$$

where g is the determinant of the metric $g_{\mu\nu}$. If we consider φ coupled to the gravitational background with only a kinetic term, then

$$S_{g,\varphi} = \frac{1}{2}\int d^4x\,\sqrt{-g}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi, \tag{5.9}$$

where we add $1/2$ in the front with the effect of changing the metric slightly and put the action in four dimensions explicitly. Comparing Equation (5.9) and Equation (5.7), it may not be obvious that it is possible to find a metric $g_{\mu\nu}$ such that the two describe the same action; however, Rivelles (2004) demonstrates that this is possible. We begin by considering a perturbation from the flat metric $g_{\mu\nu} = \eta_{\mu\nu}(1+h) + h_{\mu\nu}$ for some traceless, symmetric $h_{\mu\nu}$. Plugging into Equation (5.9), after some work we find

$$S_{g,\varphi} = \frac{1}{2}\int d^4x(\eta^{\mu\nu} - h^{\mu\nu} + h\eta^{\mu\nu})\partial_\mu\varphi\partial_\nu\varphi. \tag{5.10}$$

In this form, with care, we can find $h^{\mu\nu}$ that will make $S_{g,\varphi} = S_\varphi$:

$$h^{\mu\nu} = \theta^{\mu\alpha}F_\alpha{}^\nu + \theta^{\nu\alpha}F_\alpha{}^\mu + \frac{1}{2}\eta^{\mu\nu}\theta^{\alpha\beta}F_{\alpha\beta}, \tag{5.11}$$

with $h = 0$, so we do not need the third term in the definition of $g_{\mu\nu}$. Note that the new metric is a function of the noncommutative parameter and the field strength. For $\theta = 0$ the metric becomes appropriately flat, whereas the metric is an explicit function of the field strength, so we say that the gauge field is coupled to the gravitational background for the scalar field φ.

It would be nice if we could express the action in Equation (5.8) entirely in terms of the metric $g_{\mu\nu}$, but unfortunately this is not possible (Rivelles, 2004). It is still possible, however, to write it in the form

$$S_A = -\frac{1}{4}\int d^4x \, (F^{\mu\nu}F_{\mu\nu} + h^{\mu\nu}F_\mu{}^\rho F_{\rho\nu}), \tag{5.12}$$

so that A_μ, the gauge field, both couples to the gravitational background and the scalar field φ (Rivelles, 2004, 2003).

To confirm that this metric actually describes gravity and is not simply on some flat, nondynamical, or other manifold, we need to calculate the Riemann and Ricci tensors and the Ricci scalar. The calculations proceed mechanically to produce (Rivelles, 2003)

$$\begin{aligned} R_{\mu\nu\rho\sigma} = \frac{1}{2}\Big[& -\theta_{\alpha[\mu}\partial_{\nu]}\partial^\alpha F_{\sigma\rho} + \theta_{\rho\alpha}\partial^\alpha\partial_{[\mu}F_{\nu]\sigma} + \theta_{\sigma\alpha}\partial_\rho\partial_{[\mu}F^\alpha{}_{\nu]} \\ & + \theta^{\alpha\beta}\left(\eta_{\sigma[\mu}\partial_{\nu]}\partial_\rho F_{\alpha\beta} - \eta_{\rho[\mu}\partial_{\nu]}\partial_\sigma F_{\alpha\beta}\right)\Big], \end{aligned} \tag{5.13}$$

and

$$R_{\mu\nu} = \frac{1}{4}\left(\theta_\mu{}^\alpha\partial_\alpha\partial^\beta F_{\beta\nu} + \theta_\nu{}^\alpha\partial_\alpha\partial^\beta F_{\beta\mu} + \frac{1}{2}\eta_{\mu\nu}\theta^{\alpha\beta}\Box F_{\alpha\beta}\right), \tag{5.14}$$

and

$$R = \frac{1}{4}\theta^{\alpha\beta}\Box F_{\alpha\beta}. \tag{5.15}$$

In Equations (5.13), (5.14), and (5.15), some new notation has been introduced. The brackets around the indices $_{[]}$ (not to be confused with the commutator) mean the antisymmetrization of the expression with respect to the indices in the brackets. We can rewrite the first term of Equation (5.13), for example, as

$$\theta_{\alpha[\mu}\partial_{\nu]}\partial^\alpha F_{\sigma\rho} = \frac{1}{2}\left(\theta_{\alpha\mu}\partial_\nu\partial^\alpha F_{\sigma\rho} - \theta_{\alpha\nu}\partial_\mu\partial^\alpha F_{\sigma\rho}\right). \tag{5.16}$$

The second new notation is the D'Alembertian $\Box = \nabla^\mu\nabla_\mu$, which in this case is $\partial^\mu\partial_\mu$ since we are still using the Minkowski metric (Carroll, 1997). It can be shown that the Ricci tensor and scalar are both zero, but the Riemann curvature tensor is non-zero, meaning that the metric does not describe a flat space-time. Furthermore, since the Riemann tensor and the metric depend on the field strength $F_{\mu\nu}$, which is a dynamical variable, the curvature is dynamical. In particular, the metric describes a gravitational plane wave. Consider a solution in the absence of matter in the form $F_{\mu\nu} = k_{[\mu}F_{\nu]}$, where k_μ is a null tensor ($k^\mu k_\mu = 0$) and $k^\mu F_\mu = 0$. In this case, the Riemann tensor satisfies the plane wave equation $\partial_\alpha R_{\mu\nu\rho\sigma} = k_\alpha R_{\mu\nu\rho\sigma}$ (Rivelles, 2003, 2004). This exciting result that some noncommutative field theories are equivalent to commutative field theories with a gravitational background coupled to the gauge field, is derived only to first order in θ. One may doubt that this result will hold for expansions to higher orders in θ, but Rivelles (2004) confirms that one can still rewrite the commutative theory in terms of a non-flat metric to quadratic order in θ and that it seems the same procedure should work to all orders in θ. In fact, Banerjee (2004) uses the exact Seiberg-Witten map to rewrite the noncommutative electrodynamics action (with and without source terms) as a commutative field theory on a new nontrivial metric.

Chapter 6

More General Transformations in Noncommutative Field Theory

In addition to the Seiberg-Witten map, on the noncommutative plane another interesting transformation exists. If we consider a noncommutative field theory with parameter θ, then we can reformulate the noncommutative plane algebraically as $\mathbb{R}^3$ ((x^1, x^2, θ), where x^i are noncommuting operators which generate the plane) modded out by the equivalence relation

$$[x^i, x^j] - i\theta\epsilon^{ij} = 0, \tag{6.1}$$

where ϵ^{ij} is the *Levi-Civita* symbol which is 1 if its indices are exchanged an even number of times and -1 if its indices are exchanged an odd number of times and so is totally anti-symmetric. Examining Equation (6.1), we see that while dilation of the coordinates and rescaling of θ on their own do not leave the commutation relation unchanged, a simultaneous dilation of the coordinates and rescaling of θ that leaves Equation (6.1) unchanged should exist (this is an automorphism of the algebra). Such a transformation, then, is a symmetry of the algebra and it is possible to make the transformation such that it is a symmetry of all gauge invariant (physical) quantities. This mapping is distinct from the Seiberg-Witten map: the Seiberg-Witten map is a rescaling of θ and a simultaneous transformation of the gauge field and gauge parameter, with no dilation of the coordinates. In particular we are considering a mapping of the form

$$\varphi(x, \theta) \mapsto \rho\dot{\varphi}(x, \theta) = \varphi(\rho x, \rho^2\theta). \tag{6.2}$$

The generator of the transformation is given by

$$D = x^i \partial_i + 2\theta \partial_\theta, \tag{6.3}$$

where $\partial_\theta = \frac{\partial}{\partial \theta}$ analogously to ∂_i. We can confirm this by applying D to the left hand side of Equation (6.1) and finding that it does indeed leave the commutation relation invariant. We choose this D because it obeys the Leibnitz rule as well (Pinzul and Stern, 2004).

With a transformation of this form we have a symmetry of the action. We might imagine a noncommutative parameter that depends on time, $\theta(t)$, which in string theory corresponds to time varying magnetic fields. Assuming that the dynamics of $\theta(t)$ are specified, we can write an action of the form

$$S = \int dt d^2x \, \mathscr{L}(x,t) = \int dt d^2x \, \left|\frac{d\theta}{dt}\right| \mathscr{L}(x,\theta) = \int d\theta d^2x \, \mathscr{L}(x,\theta), \tag{6.4}$$

where we have performed a change of variables. If we suppose a massless quartic potential scalar field theory,

$$\mathscr{L} = \frac{1}{2}\partial_i\varphi\partial_i\varphi + \frac{g}{4!}\varphi^4, \tag{6.5}$$

then we can use Noether's Theorem to find the conserved current associated with the transformation's symmetry. Note that we can apply Noether's Theorem because of the algebraic reformulation; we are no longer using a noncommutative product on a space of functions on the commutative plane. For this Lagrangian density, the conserved current is

$$\begin{aligned} j^i &= -\eta^{ij}\partial_j\varphi(1+D)\varphi + \frac{1}{2}x^i\partial_j\varphi\partial_j\varphi + \frac{g}{4!}x^i\varphi^4, \\ j^\theta &= 2\theta\left(\frac{1}{2}\partial_j\varphi\partial_j\varphi + \frac{g}{4!}\varphi^4\right), \end{aligned} \tag{6.6}$$

with conservation law $\partial_i j^i + \partial_\theta j^\theta = 0$ (Pinzul and Stern, 2004). The conserved current is not on the noncommutative plane but on $\mathbb{R}^3$, the space of noncommutative planes parameterized by θ. Pinzul and Stern (2004) demonstrate that this conservation law does not correspond to a conservation law in the commutative plane by looking at the limit as θ approaches zero.

Chapter 7

Discussion

We can generalize classical field theory to noncommutative geometries where the coordinates do not commute. The consequences of moving from a commutative field theory to an analogous noncommutative field theories, while not fully understood, include: there is no longer a general formula for the equations of motion of the fields, the connection between symmetry and conservation laws is no longer clear, and it seems that, at least sometimes, moving to a noncommutative field theory is equivalent to moving to a commutative field theory with a gravitational background. In particular, we found that the noncommutative electrodynamics action is equivalent to a commutative theory on a Riemannian manifold with metric describing gravitational plane waves. Finally, we saw a more algebraic approach to noncommutative geometry where instead of using noncommutative products formed by infinite series of derivatives, we considered $\mathbb{R}^3$ modded out by an equivalence relation corresponding to a commutation relation. This meant that Noether's theorem could still be used to find the conserved current associated with symmetries.

At this time, it seems research into noncommutative geometry could benefit from a noncommutative analog of Noether's theorem that relates general symmetries of noncommutative field theories to conservation laws. Such a result may depend on the noncommutative algebra considered and a more abstract approach such as the approach taken in Sykora (2004) or Pinzul and Stern (2004). Another potential avenue of research would be to investgate how general the connection between noncommutative field theories and commutative theories on gravitational backgrounds is: perhaps, finding the form noncommutative theories may take which will lead to commutative analogs that can be expressed in terms of a non-flat metric, or finding the range of possible gravitational backgrounds that can arise from the Seiberg-Witten map of classical field theories. Another related issue is whether such a mapping is possible with nonabelian noncommutative gauge theories, a topic that was not explored in this paper.

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