An Introduction to Voting Theory

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Voting is something with which our society is very familiar. We vote in political elections on which candidates we prefer for mayor, senator, or president. We vote in committees on what actions we want our group to take. It is also a process in which we essentially take part each day. We 'vote' when we go out with our friends and need to decide which restaurant to go to or which movie to see. Voting takes place whenever a group needs to pool preferences to make an overall decision.

Unfortunately, it is not always easy to make a single choice when given the preferences of many voters. There are many different methods of voting and tallying votes. In national presidential elections, we choose our top candidate and give no more information. This is called a *plurality* voting scheme. But there are also situations in which voters might be asked to give a *full ranking* of a number of alternatives, such as in a marketing survey. If we then chose to give a number of points to these items according to where they are placed in the ranking, this would be *positional* voting. Each positional scheme is accompanied by a *weighting vector*, $\mathbf{w}$, whose entries correspond to how many points are given to a voter's first, second, ..., last choice. For simplicity, we will often normalize these vectors so that their entries range from 1 to 0, with non-increasing values. For example, in the case of plurality voting, $\mathbf{w}_{\text{plurality}} = [1, 0, \ldots, 0]$. We will also talk later about the *Borda count*, which gives a linearly decreasing number of points to a voter's preferences in order. In other words, the weighting vector for the Borda count is $\mathbf{w}_{\text{Borda}} = [1, \frac{n-2}{n-1}, \frac{n-3}{n-1}, \ldots, 0]$.

Alternatively, we could compare these items *pairwise*, and give points according to how many times a candidate beats each other candidate. In other words, candidate A wins over candidate B if A is ranked higher than B more times than B is ranked higher than A. There is also *approval* voting, which was introduced by the Venetians in the 13$^{\text{th}}$ century [6]. In this system, voters simply give out a point to each candidate of which they approve.

This leads to the question, however, of which method is the "best" – which method most accurately represents the intention of the voters? Although many methods of voting had been introduced for centuries prior, this debate made its way significantly into academia in the 18$^{\text{th}}$ century. The mathematician and philosopher Marie Jean Antoine Nicolas de Caritat Condorcet brought the pairwise method, first recorded by Ramon Llull of Spain in the 13$^{\text{th}}$ century, back under scrutiny [6]. In 1785, Condorcet published the paper *Essai sur l'application de l'analyse la probabilite des decisions rendues la pluralite des voix* (Essay on the Application of Analysis to the Probability of Majority Decisions) [2]. In this paper, he proposed the idea that if a candidate would win all head-to-head comparisons, then that candidate should win the overall election. He recognized, however, that this method would not guarantee a winner. He also discussed what has become known as the *Condorcet paradox* – that in head-to-head comparisons, a society can prefer candidate A over candidate B, candidate B over candidate

C, and candidate C over candidate A. In other words, the societal preferences in pairwise tallies may not be transitive. Because of his influence in bringing pairwise voting into focus, this method is also often referred to as the *Condorcet method*.

This paper spurred a debate between Condorcet and another French mathematician by the name of Jean Charles de Borda. Borda was known for his work on engineering in the military, and his contributions to developing the metric system. He worked with Condorcet, alongside Laplace, Lavoisier and Legendre, on the Commission of Weights and Measures (founded in 1790) [7]. Borda challenged Condorcet and Llull's method on the grounds that it was not entirely workable. Often, Condorcet's method will show cyclic preferences, and radically fail to produce a winner. Instead, Borda proposed a method developed by Nicholas of Cusa in the 15$^{\text{th}}$ century. He proposed the positional tally for n candidates which gives n points to a voter's top choice, $n - 1$ points to the second choice, and so on (as mentioned previously, we will normalize these values for our purposes). This method was also prone to producing ties, but was more likely to produce a winner. These two mathematicians continued to debate which of these methods was the most 'fair' and productive almost until Condorcet's arrest and death in 1794 [5]. Unfortunately, since their debate was mostly philosophical (despite some use of certain axiomatic and probabilistic tools), and since both systems had strengths and weaknesses, they were not able to make much headway.

To illustrate how these two tallying methods differ, let's look at an example. Let's say that we ask voters to rank three candidates – A, B, and C. To represent A beating B, we will write $A > B$ (sometimes we will simply write AB). Suppose 18 voters choose the following rankings:

votes	ranking	votes	ranking	votes	ranking
2	$A > B > C$	2	$B > A > C$	1	$C > A > B$
5	$A > C > B$	4	$B > C > A$	4	$C > B > A$

We would then collect this data in to a *profile*, **p**, which represents how many people voted for each ranking of candidates. The profile for this outcome, if we order the rankings alphabetically, is $\mathbf{p} = (2, 5, 2, 4, 1, 4)$. If we chose to use the Borda Count to tally these votes, the outcome will give 10.5, 9, and 8.5 points to A, B, and C respectively, giving an over-all ranking of $A > B > C$. However, if we chose to calculate the pairwise ranking, we will find the following tally:

points	pair	points	pair
8	$A > B$	10	$B > A$
8	$A > C$	10	$C > A$
8	$B > C$	10	$C > B$

which gives a transitive ranking of $C > B > A$. You will notice that these two methods of tallying entirely contradict each other in this case. This is what is referred to as a *paradox* because we calculated differing outcomes using two seemingly fair tallying methods.

Following Condorcet and Borda's debate, many people attempted to modify these procedures to improve upon them. For example, in 1876, mathematician Charles Ludwidge Dodgson (better known as Lewis Carroll) proposed a procedure in which a Condorcet winner will be chosen; if there is no Condorcet winner, then the candidate who needs fewest ballots to be changed to become the Condorcet winner will be chosen. John Kemeny proposed a similar system in 1959, where the winner is the candidate who requires the fewest number of rank pairs being exchanged (flipped) on voters' ballots to make that candidate win by Condorcet's rule. In 1958, Duncan Black constructed a method that blended Condorcet's and Borda's methods. Namely, the Black winner will be the Condorcet winner, unless one fails to exist – otherwise, the Borda winner will be chosen. Each of these theories provided additional strengths to Condorcet methods (i.e. we are more likely to calculate a winner), but many of them also resulted in new paradoxes and weaknesses (e.g. the resulting winner may no longer be intuitive) [8].

Another major advance in voting theory was made in the mid-20$^{\text{th}}$ century by the Nobel laureate economist Kenneth Arrow. With contributions from Blau and Murakami, he used axiomatic methods to prove that there can not exist an entirely fair voting system. More specifically, he chose four criteria, and showed that they were inconsistent. These four criteria were Universality, Independence of Irrelevant Alternatives, Citizen's Sovereignty, and Non-dictatorship [4]. *Universality* (or *unrestricted domain*) means that the election procedure should provide a full ranking (with strict preferences) for all possible sets of data. *Independence of Irrelevant Alternatives* requires that any ranking of a subset of alternatives will be unaffected by changes in rankings of other alternatives. *Citizen's Sovereignty* (or *non-imposition*) requires that the election procedure allows for all possible outcomes. *Non-dictatorship* simply means that more than one person's vote can affect the outcome. Arrow proved that if the first three criteria were true for a given election procedure, then it must be a dictatorship. This was a profound message for voting theorists, because it meant that we could not possibly construct a truly fair system. Thus, we cannot ask the question 'how can we construct a fair system'. Instead, we can only attempt to decide which system is the 'most' fair.

In the late 20$^{\text{th}}$ century, mathematician Donald Saari began to tackle this question using geometric methods. In departing from axiomatic methods, he began addressing voting structures as vector spaces. As mentioned before, we can discuss profiles and positional weightings as vectors. We can then naturally begin to study the vector spaces in which these vectors reside. This is precisely what Saari did. He decomposed profile spaces using geometric ideas in order to explain how paradoxes arise, and with what significance. By analyzing the size of various components, he could make statements as to how often certain tally methods would agree with others. Specifically, he has written quite a bit on comparing Condorcet and Borda methods.

In three major papers ([10, 11, 12]), Saari devided profile spaces into four subspaces: the *kernel* (the all ties space, generated by the all-ones vector), the *Basic* space (containing profiles for which the Borda and Condorcet tallies agree), the *Condorcet* space (containing profiles which only affect Condorcet outcomes), and the *Reversal* space (containing profiles which only affect Borda Count outcomes). One major accomplishment he has made is to offer bases for each of these spaces, which allows one to construct a profile with various kinds of conflicts or paradoxes by selecting portions of each subspace. He also showed that the positional tallying method which agrees most with pairwise tallies for full rankings is the Borda count. He has also made some steps in deciphering what happens in cases where candidates drop out of the race.

One possible natural extension on this work would be to attempt similar decompositions using algebraic techniques. Abstract algebra lends itself well to both working with the vector spaces of profiles, as well as with maps associated with them. In particular, we can describe various rankings of candidates as permutations—elements of the symmetric group. Tallying these profiles can be described as mapping their profile vectors from one space to another. One can show that these maps are $\mathbb{C}S_n$-module homomorphisms. So in particular, representation theory could prove to be quite useful. Representation theory can then help us to study the profile space in terms of the maps we use to tally the votes.

Another, perhaps more natural, extension is to move from studying fully ranked data to studying partially ranked data. It is not always entirely practical to ask for a fully ranked list from voters (for just ten candidates, this would give voters $3,620,800$ choices!). A *Partial ranking* calls for voters to place candidates into ranked sets. For example, we might ask voters to tell us, out of six candidates, their top choice, then their next two favorite, and then their three least favorite, without making any distinctions within these categories. Representation theory can help to analyze these objects as well. A significant amount of representation theory has been developed for objects that we can use to prepresent paritally ranked data [15]. Thus, these tools from algebra may prove quite useful for making generalization to this kind of data.

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