

# Matrix Representations of Knot and Link Groups

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May, 2006

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# Abstract

In the 1960s French mathematician George de Rham found a relationship between two invariants of knots. He found that there exist representations of the fundamental group of a knot into a group  $\mathcal{G}$  of upper right triangular matrices in  $\mathbb{C}$  with determinant one that is described exactly by the roots of the Alexander polynomial. I extended this result to find that the representations of the fundamental group of a link into  $\mathcal{G}$  are described by the multivariable Alexander polynomial of the link.



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# Acknowledgments

I would like to thank my adviser, Prof. Hoste, for suggesting such an interesting problem and then bearing with me while I followed every wrong track and tried to prove it impossible time and again.



# Chapter 1

## Background

### 1.1 Definition of a Knot

A *knot* is a simple closed polygonal curve in  $\mathbb{R}^3$ . Usually we think of a knot as smooth, imagining there are so many edges that we can no longer see the vertices. We can also consider multiple knots at the same time. A *link* is the finite union of disjoint knots. Thus a knot is simply a link with only one component. Two knots or links are equivalent if one can be deformed into the other by a continuous deformation of  $\mathbb{R}^3$ . Thus a knot or link is actually an equivalence class of such forms.

Throughout this work we will be dealing with knot and link diagrams. A link diagram is the projection of a link into  $\mathbb{R}^2$  such that no three points on the link project onto the same point, and no vertex projects to the same point as another point on the link. We also indicate the lower strand at each double point. This allows us to reproduce the link without ambiguity from the diagram.

We can classify diagrams by a series of three deformations called Reidemeister moves, shown in Figure 1.1. A fundamental theorem of knot theory says that two diagrams represent the same link if and only if they are related by Reidemeister moves.

A major problem in knot theory is distinguishing between different knots and links. We can make the deformations described above, but it is not obvious if two links are in fact equivalent without a lot of work. Thus, we try to come up with invariants which are independent of the diagram, this means that the invariant will be preserved through Reidemeister moves. These invariants are useful for telling us if two links are different, though they usually cannot tell us that two links are the same.

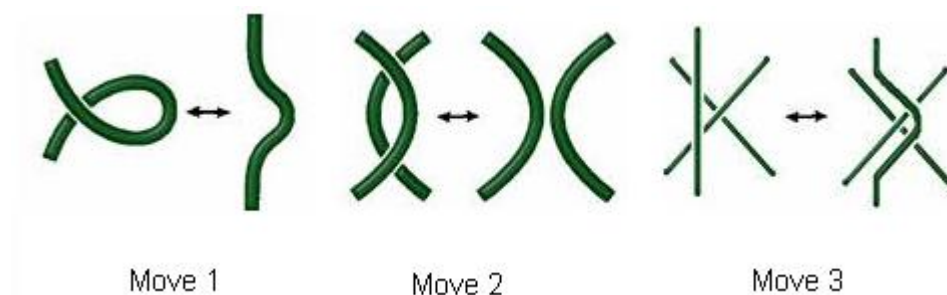


Figure 1.1: The three types of Reidemeister moves.

## 1.2 Definition of the Alexander Polynomial

The Alexander Polynomial,  $\Delta(t)$ , is an invariant of oriented links. It was first defined in 1928 by Alexander and since then has been described in multiple ways. We will begin by looking at the method described in Livingston [9]. This requires constructing a matrix from the strands and crossings of the link diagram.

A connected diagram of a link has no components that are not connected by some crossing to the other components. Given a connected link diagram we can construct the Alexander polynomial as follows:

The unknot has Alexander polynomial 1. For all other knots we arbitrarily number the crossings and separately the strands, with consecutive integers starting at one. Then we construct an  $n \times n$  matrix where  $n$  is the number of crossings and also strands. Look at each crossing,  $l$ , if it is left handed as in Figure 1.2, then in row  $l$ , enter  $1 - t$  in column  $i$ ,  $-1$  in column  $j$  and  $t$  in column  $k$ . Or if it's right handed switch the  $j$  and  $k$  entries. The remaining entries of this row will be zeros.

Removing the last row and column of this matrix produces a new  $(n - 1) \times (n - 1)$  matrix called the *Alexander matrix*. Its determinant is the *Alexander polynomial*.

**Theorem 1:** If the Alexander polynomial for a link,  $K$ , is computed using two different choices of diagrams and labeling it will differ by a factor of  $\pm t^k$  where  $k \in \mathbb{Z}$ .

**Proof:** We will demonstrate that the Reidemeister moves do not change the polynomial up to multiples of  $\pm t^{\pm i}$ .

We must examine what happens when we change the diagram. Reidemeister move I introduces one new crossing and one new strand. This

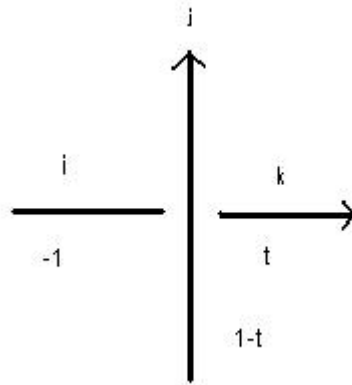


Figure 1.2: A left handed crossing labeled to make an Alexander matrix.

changes the matrix by a row and a column, it takes the crossing,  $n$ , that now has the new strand entering it and moves the entry  $x_1$  from the old  $i$  column to the new  $i + 1$  column. Then the new crossing has entries of  $y$  and  $-y$  where  $y$  is either 1 or  $t$ , which will correspond to the  $i$  and  $i + 1$  columns.

$$\begin{pmatrix} \cdots & x_2 & 0 \\ \vdots & & x_1 \\ 0 & \cdots & y & -y \end{pmatrix}$$

So if we take the new row, multiply by  $x_1/y$  and add it to row  $n$  we will attain the original  $n$  row with an extra 0 on the end.

$$\begin{pmatrix} \cdots & x_2 & 0 \\ \vdots & & x_1 & 0 \\ 0 & \cdots & y & -y \end{pmatrix}$$

This creates a column with all zeros except in the last diagonal entry, so the determinant can be taken and will change by a multiple of that entry, which will be  $\pm 1$  or  $t$ .

Reidemeister move II adds two crossings and strands, and thus two rows and columns. These will take the disrupted arc and split its entries so one remains in the same location and the other will move into one of the new columns. Also, the two rows corresponding to new crossings will both have the  $1 - t$  entry in the same column (as they are both on the same

arc) and will have one of the new columns with a -1 and a  $t$  and the other, which has the above discussed entry will have an additional either -1 or  $t$ .

$$\begin{pmatrix} \cdots & x & 0 & 0 \\ \vdots & & 0 & y \\ 0 & 1-t & t & -1 & 0 \\ 0 & 1-t & 0 & t & -1 \end{pmatrix}$$

Through a series of row and column operations we can recombine the column that was split and reduce the two new columns to zeros above the diagonal. The values in the bottom two rows are unimportant because when we take the determinant we do so following the last two columns which knock out those rows. Thus we return to the original matrix with two extra rows and columns that will add factors of  $\pm 1$  or  $\pm t$  based on their diagonal entries.

Arguments similar to those used for moves I and II show that the type III Reidemeister move does not change the Alexander polynomial except by factors of  $\pm t^{\pm i}$ .

If we change the labeling of the strands or crossings we will essentially be performing row and column operations, switching them around to fit the new configuration. If a crossing label is changed we must switch the columns, if a row label is changed we must switch the rows.  $\square$

Further reducing the Alexander matrices produces higher order Alexander polynomials which are also invariants of the knot or link.

The  $k$ th order Alexander polynomial is the greatest common divisor of the determinants of all  $n - k + 1 \times n - k + 1$  minors of the Alexander matrix, where  $n$  is the number of crossings.

We denote these polynomials as  $\Delta_k(t)$  where  $i$  is the order. In fact the Alexander polynomial  $\Delta(t)$  can also be denoted as the first order Alexander polynomial,  $\Delta_1(t)$ . It is interesting to note that  $\Delta_k \mid \Delta_{k-1}$  for all  $i$ .

### 1.3 Example Computation

Let's do an example construction of the Alexander polynomial. Looking first at a single component link we will use the trefoil knot shown in Figure 1.3. We can label the strands and crossings as indicated in the figure. Then the row corresponding to crossing 1, will be  $1 - t \quad t \quad -1$ . Thus the

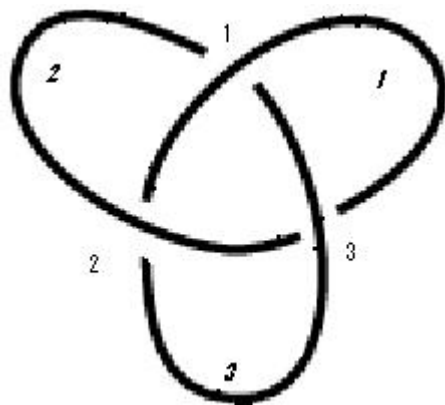


Figure 1.3: The trefoil knot.

entire matrix is

$$\begin{pmatrix} 1-t & t & -1 \\ -1 & 1-t & t \\ t & -1 & 1-t \end{pmatrix}.$$

We then strike one row and column and take the determinant:

$$\det \begin{pmatrix} 1-t & t \\ -1 & 1-t \end{pmatrix} = (1-t)^2 + t = t^2 - t + 1 = \Delta(t).$$

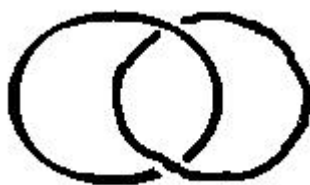


Figure 1.4: A link with two components.

Now let's look at the Hopf link with two components, shown in Figure 1.4. The Hopf link has matrix

$$\begin{pmatrix} t-1 & 1-t \\ 1-t & t-1 \end{pmatrix}.$$

Note, that when one strand is involved in multiple parts of the crossing we add together it's values. So the Alexander polynomial of the Hopf link is  $t - 1$ .

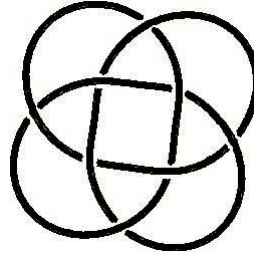


Figure 1.5: The smallest knot with nontrivial second order Alexander polynomial.

Finally, we look at a knot that has nontrivial higher order Alexander polynomials. Consider the knot  $8_{18}$  in Figure 1.5. This is the smallest knot with a nontrivial second order Alexander polynomial.

The diagram gives the matrix

$$\begin{pmatrix} t & -1 & 0 & 0 & 0 & 0 & 1-t & 0 \\ 0 & 0 & 0 & 0 & 1-t & 0 & -1 & t \\ 0 & 0 & 1-t & 0 & -1 & t & 0 & 0 \\ 1-t & 0 & -1 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-t & 0 & -1 & t & 0 \\ 0 & 1-t & 0 & -1 & t & 0 & 0 & 0 \\ 0 & -1 & t & 0 & 0 & 0 & 0 & 1-t \\ t & 0 & 0 & 0 & 0 & 1-t & 0 & -1 \end{pmatrix}.$$

Striking out one row and column gives us the polynomial

$$\Delta(t) = (t^2 - t + 1)(t^2 - 3t + 1).$$

We can also strike one row and column from the Alexander matrix to get 49 different minors, then taking the derivative of each of these and finding their greatest common divisor we get the polynomial  $\Delta_2(t) = t^2 - t + 1$ . Note that this is a factor of  $\Delta(t)$ . Also,  $\Delta_3(t) = 1$  which is a factor of  $\Delta_2(t)$ .

## 1.4 The Fundamental Group of a Link

Another invariant of knots and links is the fundamental group of the complement. If the link is embedded in  $\mathbb{R}^3$  then we look at  $\mathbb{R}^3 - K$  to find the

fundamental group of link  $K$ . The generators for the group are the equivalence classes of loops that begin and end at a fixed point in  $\mathbb{R}^3 - K$ . Two loops are equivalent if one can be continuously deformed into the other in the complement of  $K$  while keeping the same fixed point.

We can derive a presentation of the fundamental group of an oriented link from any diagram of the link using an algorithm developed by Wirtinger. The algorithm proceeds as follows:

First take the diagram and indicate an arrow under each strand, such that the arrow points in the right handed direction. Label the arrows  $a, b, c, \dots$ , as in Figure 1.6.

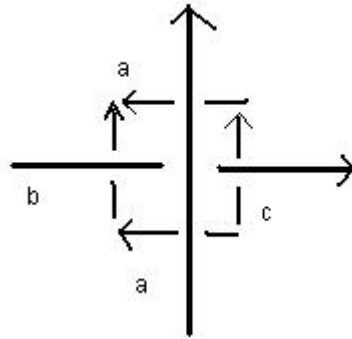


Figure 1.6: A left handed crossing with generators  $a, b, c$  and relation  $aba^{-1}c^{-1} = 1$ .

Now, when we look at any crossing we can derive a group relation. Start at the base of one of the arrows, then follow them around the crossing until you return to the starting point. Record each arrow crossed, using its inverse if it was traversed backwards. Set this value equal to 1. This is the group relation, as demonstrated in Figure 1.6.

So, the arrows are the generators of the group, and the relations are those derived from the crossings.

Example: Looking once again at the trefoil in Figure 1.7 we see that the generators are  $a, b, c$ . Looking closely at one crossing, we find the relation,  $ca = bc$ . So we get

$$\langle a, b, c \mid ca = bc, ab = bc, ca = ab \rangle,$$

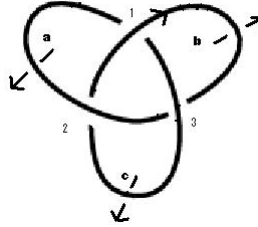


Figure 1.7: The trefoil labeled to derive the Wirtinger presentation of the fundamental group.

which can be simplified to

$$\langle a, b \mid aba = bab \rangle .$$

The fundamental group is derived from the complement of the link which tells us that it does not depend on the choice of diagram. Yet, the presentation as described above is derived from the diagram so it will change with Reidemeister moves. It is an interesting exercise to show that a different choice of diagram will produce a different presentation of the same group. The construction relying on the diagram allows us to associate a group to a link in a purely combinatorial way without any consideration of topological spaces.

Some useful properties can be derived from the construction described above.

**Proposition 2:** All Wirtinger generators assigned to a single component are conjugate.

**Proof:** Looking at any given crossing we see that it always takes the form  $ab = bc$  which can be rewritten as  $a = bcb^{-1}$ . Thus the generators,  $a$  and  $c$ , are conjugate. As we move along a single component each successive generator is conjugate to the one before it. Thus, every generator on a component is conjugate to every other one.  $\square$

**Proposition 3:** In the Wirtinger presentation of the knot group one relation is always redundant, which is to say it is a consequence of the other relations.

**Proposition 4:** If we abelianize the link group, it reduces to  $\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$  where there are as many summands as components. This is because each conjugacy class is reduced to  $a = bcb^{-1} = cbb^{-1} = c$  so they are generated by a single element with no relations, which is  $\mathbb{Z}$ .

Finally, since the link group abelianizes to a free abelian group we see that it has an infinite quotient. Thus, link groups are infinite.



## Chapter 2

# de Rham's Representation

### 2.1 Matrix Groups

A *representation* of a group,  $G$ , is a homomorphism from  $G$  into the general linear group,  $GL$ . Representations make it easier to understand what goes on in the group because matrix multiplication is well understood and fast.

Let us look at the group,  $G$ , of linear functions  $f : \mathbb{C} \rightarrow \mathbb{C}$  such that  $f(z) = az + b$  with  $a, b \in \mathbb{C}$ , and  $a \neq 0$ . We may combine two such functions by composition. If  $f_1 = a_1z + b_1$ , and  $f_2 = a_2z + b_2$ , then,

$$f_1(f_2(z)) = a_1(a_2z + b_2) + b_1 = a_1a_2z + a_1b_2 + b_1 \in G.$$

So,  $f_1 \circ f_2(z) = a_{12}z + b_{12}$ , where  $a_{12} = a_1a_2$  and  $b_{12} = a_1b_2 + b_1$ . Thus,  $G$  is closed under composition. The identity element is  $\text{id}(z) = z$ . Finally, the inverse of  $f(z) = az + b$  is  $f^{-1}(z) = z/a - ab$  which demonstrates that  $G$  is a group.

Another group of interest is the subgroup of upper triangular matrices of determinant 1 in  $GL(2, \mathbb{C})$ . We shall call this group  $\mathcal{G}$ . We can now find a homomorphism from  $G$  to  $\mathcal{G}$  and back again,  $\varphi$  and  $\psi$  respectively. Let  $\varphi : G \rightarrow \mathcal{G}$  be defined by

$$\varphi(az + b) = \begin{pmatrix} \sqrt{a} & b/\sqrt{a} \\ 0 & 1/\sqrt{a} \end{pmatrix}$$

and  $\psi : \mathcal{G} \rightarrow G$  by

$$\psi\left(\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}\right) = a^2z + ab.$$

Checking composition we find

$$\begin{aligned}\varphi(f_1)\varphi(f_2) &= \begin{pmatrix} \sqrt{a_1} & b_1/\sqrt{a_1} \\ 0 & 1/\sqrt{a_1} \end{pmatrix} \begin{pmatrix} \sqrt{a_2} & b_2/\sqrt{a_2} \\ 0 & 1/\sqrt{a_2} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{a_1 a_2} & b_2\sqrt{a_1}/\sqrt{a_2} + b_1/\sqrt{a_1 a_2} \\ 0 & 1/\sqrt{a_1 a_2} \end{pmatrix} \\ &= \varphi(f_1 \circ f_2).\end{aligned}$$

Thus  $\varphi$  holds under composition and is a homomorphism. By similar techniques we see  $\psi$  is also a homomorphism.

Now we wish to show that  $\varphi$  and  $\psi$  are inverse homomorphisms. So, we show that  $\psi(\varphi(az + b)) = \text{id}(az + b)$

$$\begin{aligned}\psi(\varphi(az + b)) &= \psi\left(\begin{pmatrix} \sqrt{a} & b/\sqrt{a} \\ 0 & 1/\sqrt{a} \end{pmatrix}\right) \\ &= az + b \\ &= \text{id}(az + b).\end{aligned}$$

By a similar argument we get that  $\varphi(\psi) = \text{id}$ . Thus,  $\psi$  and  $\varphi$  are inverse homomorphisms. A consequence of this is that  $\varphi$  is an isomorphism between  $G$  and  $\mathcal{G}$ .

Note that the elements of  $G$  are in one to one correspondence with  $\mathbb{C} \times \mathbb{C}$ , by  $az + b \rightarrow (a, b)$ , which is uncountable. Thus  $G$  is not only infinite but cannot be finitely generated.

## 2.2 Representations into $\mathcal{G}$

Let  $K$  be a knot with fundamental group,  $\Gamma$ , given by the Wirtinger presentation. As mentioned in Proposition 3, we may assume one relation is redundant, so we get  $\Gamma = \langle g_1, g_2, \dots, g_n \mid r_1, r_2, \dots, r_{n-1} \rangle$ . Now we are interested in finding a representation of  $\Gamma$  into  $\mathcal{G}$ . In particular, we want some homomorphism,  $\rho : \Gamma \rightarrow \mathcal{G}$ . Such a homomorphism is determined by where it sends the generators. So the question becomes, can we assign matrices to  $g_i$  such that the relations  $r_j$  all hold?

A representation  $\rho : \Gamma \rightarrow \mathcal{G}$  is abelian if the image of  $\Gamma$  in  $\mathcal{G}$  is abelian.

We can consider the trivial case, that in which  $\rho(g_i) = A$  for some  $A \in \mathcal{G}$  and all  $i$ . The image is an abelian group, specifically the cyclic subgroup of  $\mathcal{G}$  generated by  $A$ . In fact, this is the only way to get an abelian representation.

**Lemma 5:** If  $\Gamma = \langle g_1, g_2, \dots, g_n \mid r_1, r_2, \dots, r_{n-1} \rangle$  is the Wirtinger presentation of a knot group, and  $\rho : \Gamma \rightarrow \mathcal{G}$  is a representation, then  $\rho$  is abelian iff  $\rho(g_i) = A, \forall i$  and some matrix  $A \in \mathcal{G}$ .

**Proof:** Suppose  $\rho$  is an abelian representation. We can consider the relations coming from a single crossing. If we have the matrices  $A, B$ , and  $C$  representing the respective strands, then  $A = BCB^{-1}$ . Since  $\rho$  is an abelian representation;  $A = BB^{-1}C = C$ . Thus, every generator must be sent to the same matrix.

Finally, as already noted, if all the generators are sent to the same matrix, then the representation is abelian.  $\square$

Such a representation loses all information from the original knot group, so we are not interested in abelian representations. Consequently, we need to come up with a collection of matrices in  $\mathcal{G}$  that do not generate an abelian subgroup. Let's try matrices with diagonal entries of 1.

Note that any such matrices commute:

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

Similarly, upper triangular matrices with  $-1$  on the diagonal commute. Thus, if we do not want an abelian representation we must avoid sending all of the generators to matrices with 1 or  $-1$  on the diagonal.

So, let's think some more about the diagonal entries, since we know they cannot all be  $\pm 1$ . The relations from  $\Gamma$  require that the matrices be conjugate, what does this do to the eigenvalues?

**Lemma 6:** Conjugate matrices have the same eigenvalues.

**Proof:** Let  $A, B, C$  be matrices such that  $A = BCB^{-1}$ . Then

$$\begin{aligned} \det(A - \lambda I) &= \det(BCB^{-1} - \lambda I) \\ &= \det(BCB^{-1} - \lambda B I B^{-1}) \\ &= \det(B(C - \lambda I)B^{-1}) \\ &= \det B \det(C - \lambda I) \det B^{-1} \\ &= \det B \det B^{-1} \det(C - \lambda I) \\ &= \det(C - \lambda I). \end{aligned}$$

Thus, the eigenvalues of conjugate matrices are the same.  $\square$

It follows that all of our matrices will have the same eigenvalues, and thus the same entries on the diagonal. Thus none of the generators may be sent to a matrix with eigenvalues of 1 or  $-1$ . Since the matrices have determinant 1, we know that the eigenvalues must be reciprocal. So, they will be  $m$  and  $1/m, m \in \mathbb{C}$ .

**Lemma 7:** Within a conjugacy class of  $\mathcal{G}$ , the eigenvalues have fixed position.

**Proof:** Let us look at one matrix within our group,  $A = \begin{pmatrix} m & a \\ 0 & 1/m \end{pmatrix}$ . Using a generic element,  $C = \begin{pmatrix} x & c \\ 0 & 1/x \end{pmatrix}$ , we can get from  $A$  to any other element,  $CAC^{-1}$ , within the conjugacy class.

$$\begin{aligned} CAC^{-1} &= \begin{pmatrix} x & c \\ 0 & 1/x \end{pmatrix} \begin{pmatrix} m & a \\ 0 & 1/m \end{pmatrix} \begin{pmatrix} 1/x & -c \\ 0 & x \end{pmatrix} \\ &= \begin{pmatrix} m & x^2a + xcm - xc/m \\ 0 & 1/m \end{pmatrix}. \end{aligned}$$

This construction demonstrates that the diagonal elements of  $A$  are never going to be changed by conjugation.  $\square$

Thus, the generators of our knot group are all taken to matrices of the form  $\begin{pmatrix} m & y \\ 0 & 1/m \end{pmatrix}$  with some fixed  $m \in \mathbb{C}$ ,  $m \neq \pm 1$  and some varying  $y \in \mathbb{C}$ .

Two representations,  $\rho$  and  $\tau$ , both from  $\Gamma \rightarrow \mathcal{G}$ , are *conjugate* if there exists an inner automorphism,  $f : \mathcal{G} \rightarrow \mathcal{G}$ , given by  $f(x) = AxA^{-1}$ ,  $A \in \mathcal{G}$  such that  $f \circ \rho = \tau$ .

If we have  $\rho(g_i) = \begin{pmatrix} m & a_i \\ 0 & m^{-1} \end{pmatrix}$ , we can change  $a_i$  through conjugation. Recalling that conjugation by an arbitrary matrix  $\begin{pmatrix} x & c \\ 0 & 1/x \end{pmatrix}$  creates the new upper right entry,  $z$ ,

$$z = x^2a_i + xcm - xc/m.$$

We can choose a value for  $z$ , and we can find the corresponding values of  $x$  and  $c$ . This determines the element by which we will conjugate to fix one  $a_i$  in the representation.

**Proposition 8:** If  $\rho$  and  $\tau$  are two non-abelian representations of  $\Gamma$  into  $\mathcal{G}$  such that  $\rho(g_i) = \begin{pmatrix} m & a_i \\ 0 & 1/m \end{pmatrix}$ , and  $\tau(g_i) = \begin{pmatrix} m & b_i \\ 0 & 1/m \end{pmatrix}$ , with  $a_1 = b_1 = 0$ , then  $\rho$  and  $\tau$  are conjugate iff there exists  $d^2 \neq 0$  such that  $a_i = d^2b_i$  with  $d \in \mathbb{C}$ .

**Proof:** [sketch] Looking at the conjugacy relation of upper right entries we know that to reach a new entry,  $z$ , we get the relation

$$z = x^2a_i + xcm - xc/m$$

in the upper right entry. But, setting  $z = 0$  we get:

$$\begin{aligned} 0 &= x^2 a_i + xc(m - 1/m) \\ c(1/m - m) &= xa_i \\ c/x &= a_i/(1/m - m). \end{aligned}$$

So, we can preserve the relation, and change  $c$  and  $x$  by some constant multiple  $d$ . But, when we make this change, every other element  $a_i$  in the representation will be scaled by

$$z_i = d^2 x^2 a_i + dx d c m - dx d c / m = d^2 z.$$

Thus, any two representations with  $a_1 = b_1 = 0$  and  $a_i = d^2 b_i$  with  $d \in \mathbb{C}$  are conjugate.  $\square$

### 2.3 Example Construction

Let us first do a simple example of the representation of a fundamental knot group. Looking at the trefoil knot in Figure 1.3 we can label the arcs  $A, B, C$ , where

$$A = \begin{pmatrix} m & a \\ 0 & 1/m \end{pmatrix}, B = \begin{pmatrix} m & b \\ 0 & 1/m \end{pmatrix}, \text{ and } C = \begin{pmatrix} m & c \\ 0 & 1/m \end{pmatrix}.$$

We know that the fundamental group is  $\langle A, B \mid ABA^{-1} = B^{-1}AB \rangle$ , so, applying the relation to our matrix generators we find

$$\begin{aligned} ABA^{-1} &= \begin{pmatrix} m & a \\ 0 & 1/m \end{pmatrix} \begin{pmatrix} m & b \\ 0 & 1/m \end{pmatrix} \begin{pmatrix} 1/m & -a \\ 0 & m \end{pmatrix} \\ B^{-1}AB &= \begin{pmatrix} 1/m & -b \\ 0 & m \end{pmatrix} \begin{pmatrix} m & a \\ 0 & 1/m \end{pmatrix} \begin{pmatrix} m & b \\ 0 & 1/m \end{pmatrix}. \end{aligned}$$

Giving us

$$\begin{pmatrix} m & bm^2 + a - am^2 \\ 0 & 1/m \end{pmatrix} = \begin{pmatrix} m & b + a/m^2 + b/m^2 \\ 0 & 1/m \end{pmatrix}.$$

So, if  $bm^2 + a - am^2 = b + a/m^2 - b/m^2$ , the relation holds. This equation is equivalent to

$$(b - a)(m^4 - m^2 + 1) = 0,$$

which is equivalent to  $(b - a)\Delta(m^2) = 0$ . Since we know  $a \neq b$  (else we have an abelian representation), we require that  $m^2$  be a root of the Alexander polynomial.

Thus for the trefoil, the non-abelian representations to  $\mathcal{G}$  correspond to the roots of the Alexander polynomial.

## 2.4 Eigenvalues and Roots

We can now claim that the eigenvalues of matrices in the representation are always derived from the roots of the Alexander Polynomial.

**Theorem 9** (De Rham): The non-abelian representations of a knot group,  $\Gamma_K$  into  $\mathcal{G}$ , correspond to the roots of the Alexander polynomial,  $\Delta(t)$ . For each root,  $\lambda$ , there are  $2k_\lambda$  representations, up to conjugation, when  $k_\lambda$  is the largest  $k$  such that  $\Delta_k(\lambda) \neq 0$ .

**Proof:** Let us examine the Livingston method of constructing the Alexander Polynomial. Given any crossing,  $i$  (Figure 1.2), we get the  $i$ th row of the Alexander matrix.

$$\begin{pmatrix} \cdots & & & & \\ 1-t & t & -1 & 0 & \\ \cdots & & & & \\ \cdots & & & & \end{pmatrix}$$

Similarly, looking at the fundamental knot group we can once again look at the  $i$ th crossing. Applying the matrix representation with every element of the form  $\begin{pmatrix} \sqrt{t} & x \\ 0 & 1/\sqrt{t} \end{pmatrix}$ , where  $t$  is fixed across all matrices, and  $x$  varies between matrices, we find the relations. Then, for the  $i$ th crossing we get the relation:

$$AB = CA,$$

which yields:

$$\begin{aligned} \begin{pmatrix} \sqrt{t} & x_1 \\ 0 & 1/\sqrt{t} \end{pmatrix} \begin{pmatrix} \sqrt{t} & x_2 \\ 0 & 1/\sqrt{t} \end{pmatrix} &= \begin{pmatrix} \sqrt{t} & x_3 \\ 0 & 1/\sqrt{t} \end{pmatrix} \begin{pmatrix} \sqrt{t} & x_1 \\ 0 & 1/\sqrt{t} \end{pmatrix} \\ \begin{pmatrix} t & x_2\sqrt{t} + x_1/\sqrt{t} \\ 0 & 1/t \end{pmatrix} &= \begin{pmatrix} t & x_1\sqrt{t} + x_3/\sqrt{t} \\ 0 & 1/t \end{pmatrix}. \end{aligned}$$

This demonstrates that, so long as the upper right entries are equal, the representation will hold. So,

$$\begin{aligned} x_2\sqrt{t} + x_1/\sqrt{t} &= x_1\sqrt{t} + x_3/\sqrt{t} \\ x_2t + x_1 &= x_1t + x_3 \\ x_1(1-t) + x_2t - x_3 &= 0. \end{aligned}$$

Looking closely at this result we realize that it is identical to the  $i$ th row of

the Alexander matrix multiplied by a vector with  $a, b, c$ , and is equal to 0.

$$\begin{pmatrix} \dots & & & & \\ 1-t & t & -1 & 0 & \\ \dots & & & & \\ \dots & & & & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The equation above includes the matrix which gives us the Alexander matrix but it has been found by applying our representation to the fundamental group of the knot. To actually get the Alexander polynomial from the matrix we must be able to eliminate any one row and any one column.

We know by the construction of the fundamental group that any one of the relations is always a result of the others. So, we can eliminate one row. Also, using conjugate representations we can fix one  $x$  to 0. Since we are looking for a non-abelian representation we know that not all elements are the same and conjugating one element to zero scales all remaining elements, so not all  $x_i$  will be sent to 0. When we multiply the matrix with the vector of  $x$  values, the column corresponding to the 0 entry will simply be taken to zero. So, we can remove that column and entry from the vector.

Now we have the Alexander matrix,  $A_{\Delta(t)}$ , but we must look at the rest of the equation to figure out what the  $t$  values can be. Since we are multiplying by a nonzero vector, and since we attain a zero vector,  $A_{\Delta(t)}$  must be linearly independent. So,  $\det(A_{\Delta(t)}) = 0 = \Delta(t)$  must hold and  $t$  must be a root of the Alexander polynomial or the relation will not hold. Thus the diagonal entries,  $\sqrt{t}, 1/\sqrt{t}$ , rely on the Alexander polynomial to make a representation of the fundamental knot group.

For each of these diagonal entries we can find the dimension of the corresponding null space of the matrix by looking at the  $k$ th order Alexander polynomials. If  $\Delta_i(t) = 0$  then the matrix is linearly dependent, so there must be at least one redundant column. Thus the  $k$  with nonzero  $\Delta_k(t)$  gives us the number of linearly dependent columns,  $k$ , and thus the dimension of the null space. Since sending one of our  $x_i$  to zero by conjugation still gives us freedom to scale the vector, we only need to get each element in the basis of the null space to find the number of conjugacy classes of the representation. Thus, we have  $k$  distinct conjugacy classes for each diagonal entry, and there are 2 of these for each root of  $\Delta(t)$ . So, we get exactly  $\sum_{\lambda} 2 * k_{\lambda}$  distinct conjugacy classes, where  $\lambda$  are the roots of the Alexander polynomial, and  $k_{\lambda}$  are the largest values for which we have nonzero  $\Delta_k(\lambda)$ .  $\square$



## Chapter 3

# Different Approaches

### 3.1 Further consideration of the Alexander Polynomial

Recall that there are multiple methods of deriving the Alexander polynomial. We have already looked at the Livingston method. Now let us look at Alexander's original method.

Using the Alexander method, we label the regions of the diagram with  $r_i$  starting with  $r_0$  in the region surrounding the knot. Then, we can arbitrarily label the crossings 1 through  $n$  and the components  $K_1$  through  $K_m$ . Once again we get relations from each crossing that can be combined to produce a matrix. The matrix will have rows corresponding to crossings and columns corresponding to regions, making an  $n \times (n + 2)$  matrix. The relation from any  $i$ th crossing (Figure 3.1) is:

$$t_j r_k - t_j r_l + r_p - r_q,$$

where the under-crossing strand is part of the  $j$ th component. Note that the  $r_k$  and  $r_l$  regions will always be to the right of the under-crossing strand and  $r_l$  will be situated counterclockwise from  $r_k$ . We can then delete any two columns that represent regions separated by one strand,  $h$ . The Alexander polynomial is the determinant of our new matrix divided by  $(t_h - 1)$ .

The process above produces the multivariable Alexander polynomial,  $\Delta(t_1, \dots, t_m)$ . As with the single variable Alexander polynomial, the multivariable polynomial is invariant up to multiplication by  $\pm t_j^{\pm 1}$  depending on the choice of columns to delete and choice of diagram. If we set all of the  $t_j = t$ , and multiply by  $(t - 1)$ , we get the single variable Alexander polynomial,  $\Delta(t)$ , discussed earlier.

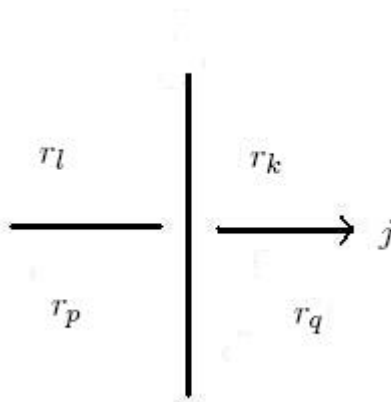


Figure 3.1: A crossing labeled to use Alexander's method of finding the Alexander polynomial.

### 3.2 Another Example Computation

Let's do a couple of examples of the Alexander method of finding the Alexander polynomial.

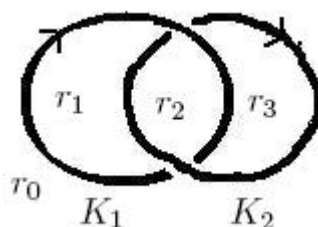


Figure 3.2: The Hopf link.

First we will look at the Hopf link (Figure 3.2), which we previously examined using the Livingston method. Using this method we get the matrix:

$$\begin{pmatrix} t_2 & -t_2 & 1 & -1 \\ t_1 & -1 & 1 & -t_1 \end{pmatrix}.$$

We can eliminate the first two columns of this matrix. These columns cor-

respond to adjacent regions separated by component 1. This gives:

$$\begin{pmatrix} 1 & -1 \\ 1 & -t_1 \end{pmatrix}.$$

The determinant of this matrix, divided by  $(t_1 - 1)$ , gives us the multivariable Alexander polynomial, 1. Furthermore, if we set  $t_1 = t_2 = t$ , we get the single variable Alexander polynomial from before,  $t - 1$ .

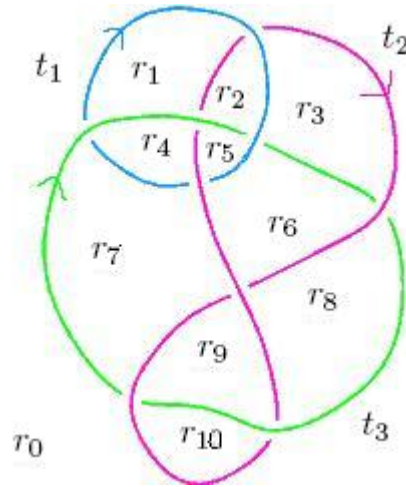


Figure 3.3: The link L9n23.

Now let's look at a more complicated example, the three component non-alternating link L9n23 (Figure 3.3). For this link we get the matrix:

$$\begin{pmatrix} t_2 & -t_2 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_1 & -1 & 0 & 0 & 1 & 0 & 0 & -t_1 & 0 & 0 & 0 \\ 0 & t_2 & -1 & 0 & -t_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -t_3 & t_3 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -t_1 & t_1 & 0 & 0 & 0 \\ t_3 & 0 & 0 & -t_3 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -t_2 & t_2 & 0 \\ t_3 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -t_3 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & t_2 & -t_2 \end{pmatrix}.$$

We can delete columns  $r_0$  and  $r_1$  by crossing component 1. This matrix produces the Alexander polynomial

$$\Delta(t_1, t_2, t_3) = -t_3 t_2^2 t_1 - t_1 t_2 t_3^2 + t_1 t_2^2 t_3^2 + t_2 - 1 + t_3.$$

### 3.3 The Dehn Presentation

Note that we earlier looked at the Wirtinger presentation of the fundamental group of the complement of our link. Another useful algorithm creates the Dehn presentation of the fundamental group. This presentation mirrors Alexander's method of finding the Alexander polynomial by relying on looking at regions of the diagram instead of looking at the strands of the knot or link.

To find the Dehn presentation we first label each region with  $r_i$  as we did in Section 3.1. Then looking at each crossing individually we find the relation by marking the left side of the under-crossing strand then setting the left region times the inverse of the right region equal across the crossing. For example looking at the crossing in Figure 3.1 we get the relation

$$r_k r_q^{-1} = r_l r_p^{-1}.$$

Thus we find that the regions are the generators and the relations correspond to the crossings as before. In this presentation we use the convention that the outside region,  $r_0$ , is the identity element. Also, we find that at least one of the remaining generators will always be redundant.

This method derives from thinking of the loops that generate the fundamental group as traveling down through one region,  $r_i$ , then back up through the other,  $r_o$ .

Let's look at an example to become familiar with this method.

Example: Looking once again at the Hopf link Figure 3.2 we initially get two relation and four generators.

$$\begin{aligned} r_0 r_3^{-1} &= r_1 r_2^{-1} \\ r_0 r_1^{-1} &= r_3 r_2^{-1} \end{aligned}$$

Then set  $r_0 = 1$  to get

$$\begin{aligned} r_3^{-1} &= r_1 r_2^{-1} \\ r_1^{-1} &= r_3 r_2^{-1} \end{aligned}$$

which gives us  $r_1 = r_3^{-1} r_2$ . Thus we present the group as

$$\langle r_2, r_3 \mid r_2 r_3 = r_3 r_2 \rangle$$

It is important to remember that this presentation is isomorphic to the presentation described by Wirtinger. So, this is a presentation of the fundamental group of the complement space of the link and is guaranteed to be invariant to changes in diagram. However, the Dehn generators are not

identical to Wirtinger generators and we lose some of the relations between them. The Wirtinger presentation has the property that all generators associated to the same component are conjugate, but we do not have this property in the Dehn presentation.

### 3.4 The Degree

We can describe the regions of the knot as having a *degree*, notated  $d(r_i) = (d_1, \dots, d_m)$ . The degree of any region in a link diagram is defined by an  $m$ -tuple, where there are  $m$  total components. We define the outer region,  $r_0$ , to have degree 0, in all components. To find the degree of another region if we pass from the right of the strand to its left we add one to that component's degree. If we pass left to right we subtract one.

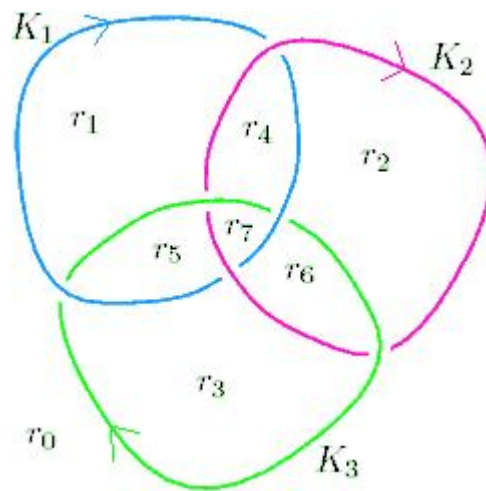


Figure 3.4: The Borromean rings with labeled regions and components.

Example: Looking at the Borromean Rings, Figure 3.4, we can identify the degree of each region. They will all be 3-tuples because there are three

components. The degrees are:

$$\begin{aligned}d(r_0) &= (0, 0, 0) \\d(r_1) &= (-1, 0, 0) \\d(r_2) &= (0, -1, 0) \\d(r_3) &= (0, 0, -1) \\d(r_4) &= (-1, -1, 0) \\d(r_5) &= (-1, 0, -1) \\d(r_6) &= (0, -1, -1) \\d(r_7) &= (-1, -1, -1).\end{aligned}$$

## Chapter 4

# The Multivariable Representation

### 4.1 Link Representations and the Multivariable Alexander Polynomial

Recall de Rham found representations of the fundamental group of knots into  $\mathcal{G}$  that were described exactly by the roots of the Alexander polynomial. I am looking to find a similar relationship between representations of the fundamental group of a link into  $\mathcal{G}$  and the multivariable Alexander polynomial.

It is fairly easy to find that roots of the Alexander polynomial will give us representations. If we use the Dehn presentation and Alexander's method to produce the multivariable Alexander polynomial we can find the representation that we want. Let us suppose that  $t_1, t_2, \dots, t_m$  are all nonzero solutions to the multivariable Alexander polynomial,  $\Delta(t_1, t_2, \dots, t_m) = 0$ .

Looking at  $\pi(K)$  we get generators  $r_i$  which correspond to regions of the link. First, we define  $\mathbf{t}^{d(r_i)}$  to be the product of  $t_j^{d_j}$ , complex numbers corresponding to each component raised to the degree associated with that component. Then we define the representation

$$\rho(r_i) = \begin{pmatrix} \mathbf{t}^{-d(r_i)/2} & R_i \mathbf{t}^{d(r_i)/2} \\ 0 & \mathbf{t}^{d(r_i)/2} \end{pmatrix}.$$

This will only be a valid representation if the relations formed at each crossing hold. So it must be true that

$$\rho(r_i)\rho(r_j^{-1}) = \rho(r_k)\rho(r_l^{-1})$$

for each crossing. So, we must get

$$\begin{aligned} \rho(r_i)\rho(r_j^{-1}) &= \begin{pmatrix} \mathbf{t}^{-d(r_i)/2} & R_i\mathbf{t}^{d(r_i)/2} \\ 0 & \mathbf{t}^{d(r_i)/2} \end{pmatrix} \begin{pmatrix} \mathbf{t}^{d(r_j)/2} & -R_j\mathbf{t}^{d(r_j)/2} \\ 0 & \mathbf{t}^{-d(r_j)/2} \end{pmatrix} \\ \rho(r_k)\rho(r_l^{-1}) &= \begin{pmatrix} \mathbf{t}^{-d(r_k)/2} & R_k\mathbf{t}^{d(r_k)/2} \\ 0 & \mathbf{t}^{d(r_k)/2} \end{pmatrix} \begin{pmatrix} \mathbf{t}^{d(r_l)/2} & -R_l\mathbf{t}^{d(r_l)/2} \\ 0 & \mathbf{t}^{-d(r_l)/2} \end{pmatrix}, \end{aligned}$$

such that

$$\begin{pmatrix} t_p^{-1/2} & R_i t_p^{1/2} - R_j t_p^{-1/2} \\ 0 & t_p^{1/2} \end{pmatrix} = \begin{pmatrix} t_p^{-1/2} & R_k t_p^{1/2} - R_l t_p^{-1/2} \\ 0 & t_p^{1/2} \end{pmatrix}.$$

The only requirement is that

$$R_i t_p^{1/2} - R_j t_p^{-1/2} = R_k t_p^{1/2} - R_l t_p^{-1/2}$$

which can be rewritten as

$$R_i t_p - R_k t_p + R_l - R_j = 0.$$

This correlates exactly with the row in the Alexander matrix. Thus, looking at all of the crossings we find a system of equations that must be satisfied for the representation to hold and this system exactly reproduces the matrix that gives us the Alexander polynomial.

Now, since our  $t_i$  are solutions to  $\Delta(t_1, t_2, \dots, t_n) = 0$  we know that they must produce a nontrivial solution to this system of equations. Note, if any  $t_i = 0$  then it will not be a valid eigenvalue because  $t_i^{-1}$  is undefined, so we do not include these solutions. So, reducing this matrix by two columns as in the production of the Alexander polynomial, we will find the linearly independent system that gave us this solution. In fact the kernel of this matrix will give us the values for the  $R_i$ , with the two columns we eliminated having  $R_i = 0$ .

Thus, we know that nonzero solutions to the multivariable Alexander polynomial give us representations of the fundamental group of a link into  $\mathcal{G}$  and the kernel of the matrix gives us the  $R_i$ .

## 4.2 Constraints on the Representation

Now we have established that nonzero roots of the multivariable Alexander polynomial correspond to representations of the fundamental group of

the link. But, we have to ask, do these roots give us every possible representation?

First, let us consider the most general representation. So, let us take all generators  $r_i \in \pi(K)$  to elements  $\rho(r_i) \in \mathcal{G}$  by:

$$\rho(r_i) = \begin{pmatrix} m_i & a_i \\ 0 & 1/m_i \end{pmatrix}$$

with  $m_i \neq 0$ .

Now, we must recall that we are uninterested in abelian representations.

**Lemma 10:** Let  $K$  be a link and  $\pi(K)$  be the Dehn presentation of its fundamental group. Then  $\rho(\pi(K))$  is abelian iff the eigenvalues are all  $\pm 1$  or the generators all have equal values for the fraction

$$\frac{R_i m_i^{-1}}{m_i - m_i^{-1}}$$

with the exception of the positive and negative identities.

**Proof:** Let  $\rho(r_i)$  and  $\rho(r_j)$  be two arbitrary elements in  $\rho(\pi(K))$ . Then by definition they are abelian if and only if

$$\rho(r_i)\rho(r_j) = \rho(r_j)\rho(r_i)$$

so that

$$\begin{pmatrix} m_i & R_i m_i^{-1} \\ 0 & m_i^{-1} \end{pmatrix} \begin{pmatrix} m_j & R_j m_j^{-1} \\ 0 & m_j^{-1} \end{pmatrix} = \begin{pmatrix} m_j & R_j m_j^{-1} \\ 0 & m_j^{-1} \end{pmatrix} \begin{pmatrix} m_i & R_i m_i^{-1} \\ 0 & m_i^{-1} \end{pmatrix} \\ \begin{pmatrix} m_i m_j & m_i m_j^{-1} R_j + m_i^{-1} m_j^{-1} R_i \\ 0 & m_i^{-1} m_j^{-1} \end{pmatrix} = \begin{pmatrix} m_i m_j & m_i^{-1} m_j R_i + m_i^{-1} m_j^{-1} R_j \\ 0 & m_i^{-1} m_j^{-1} \end{pmatrix}.$$

This shows us that the representation will be abelian when

$$m_i m_j^{-1} R_j + m_i^{-1} m_j^{-1} R_i = m_i^{-1} m_j R_i + m_i^{-1} m_j^{-1} R_j.$$

This can be rewritten as

$$\frac{R_i m_i^{-1}}{m_i - m_i^{-1}} = \frac{R_j m_j^{-1}}{m_j - m_j^{-1}}.$$

Thus, if this relation holds for all elements the relation will be abelian. But elements with  $\pm 1$  in the eigenvalue positions do not fit here, so we must consider what happens in these cases. Another way to write this relation is

$$R_j m_j^{-1} (m_i - m_i^{-1}) = R_i m_i^{-1} (m_j - m_j^{-1}).$$

So, this demonstrates that if the eigenvalues equal  $\pm 1$  for all generators the relation will hold as well. The final option is when only some of the generators have the one eigenvalues. Note, that  $m_i$  can never equal 0, because then  $m_i^{-1}$  is undefined. So the only way to make the abelian relation is for  $R_k = 0$  for all  $m_k = \pm 1$ , thus allowing the relation to hold for all other elements.

So, the only possible abelian representations have either every non-identity generator upholding the relation or every generator having eigenvalues of  $\pm 1$ .  $\square$

Now that we know what abelian representations look like we can look at the most general possible non-abelian representation and find out what constraints exist on these representations.

We know that the relations imposed at each crossing must hold in  $\mathcal{G}$ . Since  $\mathcal{G}$  is composed of two by two upper right triangular matrices we will get three equations from each relation, an upper left, upper right, and lower right. Every crossing creates a relation of the form

$$r_k r_l^{-1} = r_p r_q^{-1}.$$

When we take this to  $\mathcal{G}$  we get

$$\begin{aligned} \begin{pmatrix} m_k & a_k \\ 0 & 1/m_k \end{pmatrix} \begin{pmatrix} 1/m_l & -a_l \\ 0 & m_l \end{pmatrix} &= \begin{pmatrix} m_p & a_p \\ 0 & 1/m_p \end{pmatrix} \begin{pmatrix} 1/m_q & -a_q \\ 0 & m_q \end{pmatrix} \\ \begin{pmatrix} m_k m_l^{-1} & a_k m_l^{-1} - m_k a_l \\ 0 & m_l m_k^{-1} \end{pmatrix} &= \begin{pmatrix} m_p m_q^{-1} & a_p m_q^{-1} - m_p a_q \\ 0 & m_q m_p^{-1} \end{pmatrix} \end{aligned}$$

This gives us the three relations

$$m_k m_l^{-1} = m_p m_q^{-1} \tag{4.1}$$

$$a_k m_l^{-1} - m_k a_l = a_p m_q^{-1} - m_p a_q \tag{4.2}$$

$$m_l m_k^{-1} = m_q m_p^{-1} \tag{4.3}$$

As noted above if Equation 5.1 is true then Equation 5.3 is trivially true, because it is the inverse of Equation 5.1. So we must find some constraint that will force these to be true.

Let us start with the upper left entry. We know from our construction that  $m_i \in \mathbb{C}$  where  $m_i$  is any arbitrary element, so all  $m_i$  commute. We can think about following a single component around the link and find a relation imposed upon the upper left entry.

**Lemma 11:** Let  $K$  be a link with Dehn presentation  $\pi(K)$ . At any two points along a component of  $K$  we can consider the regions on either side of  $K$  as  $s_i$  on the right and  $l_i$  on the left. Then we find that  $m_{l_1}m_{s_1}^{-1} = m_{l_2}m_{s_2}^{-1}$ .

**Proof:** The regions surrounding any two points on  $K$  will be connected by a series of crossing relations. Since there are only four types of crossings we can learn what the relations across each type of crossing are and use this to find a relation between all regions.

Let us look at the four possible crossings that a strand can be involved in. If it is the under-strand as in the generic example above (Figure ??), we do not have to worry about the orientation of the over strand, because of the construction of the Dehn presentation. Thus, when we are looking at an under-strand we find  $m_k m_l^{-1} = m_p m_q^{-1}$ .

If we look at the case when the important strand is on top we must consider two cases. First we look at the under-strand traveling left to find  $m_p m_k^{-1} = m_q m_l^{-1}$  which can be rewritten  $m_p m_q^{-1} = m_k m_l^{-1}$  because the  $m_i$  commute. The final case is when the under-strand travels right, which gives us  $m_k m_p^{-1} = m_l m_q^{-1}$  yielding  $m_k m_l^{-1} = m_p m_q^{-1}$ .

Thus, since every type of crossing yields the same relation we know that at any two points along  $K$  the regions will be connected by a series of the same relation giving us  $m_{l_1} m_{s_1}^{-1} = m_{l_2} m_{s_2}^{-1}$ .  $\square$

We can, in fact, define this value that we found in Lemma 10. We can define  $t_i^{-1/2} = m_l m_s^{-1}$  where  $i$  indexes the component and  $m_l$  and  $m_s$  correspond to the left and right sides of any point along the component as described above.

**Lemma 12:** If  $r_i$  is any region of a link diagram,  $K$ , with  $m$  components then for  $\rho(r_i) \in \mathcal{G}$  we get  $m_i = \mathbf{t}^{-d(r_i)/2} = t_1^{-d_1(r_i)/2} t_2^{-d_2(r_i)/2} \dots t_m^{-d_m(r_i)/2}$ .

**Proof:** Let  $K$  be a link and let  $r_i$  be any region in the diagram of  $K$ . We can pick some path from  $r_0$  to  $r_i$ . We know that  $m_0 = 1$  because  $\rho(r_0)$  is the identity matrix.

Note that every time we cross a strand, on component  $K_j$ , we get the relation in Lemma 10:  $m_l m_s^{-1} = t_j^{-1/2}$ . So, if we cross from left to right we get that  $m_s = m_l t_j^{1/2}$  which is to say the degree is changed by a factor of  $t_j^{1/2}$ . If we cross from right to left we get  $m_l = m_s t_j^{-1/2}$ , so the degree is increased by a factor of  $t_j^{-1/2}$ .

This means that every time the path from  $r_0$  to  $r_i$  crosses a strand it is incremented by some factor of  $t_j^{-1/2}$ . In fact the factor will change following the same pattern as the degree. Also, no factors outside of the  $t_j$  will be introduced because  $m_0 = 1$  and crossing each strand must change it only

by a factor of  $t_j^{-1/2}$ . Thus, we can write each  $m_i$  as

$$m_i = \mathbf{t}^{-d(r_i)/2}.$$

□

We get a couple of nice corollaries from this lemma.

**Corollary 13:** Given four regions of a link,  $r_i, r_j, r_k$ , and  $r_l$ , with  $d(r_i) - d(r_j) = d(r_k) - d(r_l)$ ,

$$\frac{m_i}{m_j} = \frac{m_k}{m_l}.$$

**Proof:** Let  $K$  be a link with the four regions described above. We know that

$$d(r_i) - d(r_j) = d(r_k) - d(r_l) = (d_1, d_2, \dots, d_n),$$

such that

$$m_i = m_j t_1^{-d_1/2} t_2^{-d_2/2} \dots t_n^{-d_n/2}$$

and

$$m_k = m_l t_1^{-d_1/2} t_2^{-d_2/2} \dots t_n^{-d_n/2}.$$

So

$$m_i/m_j = m_k/m_l = t_1^{-d_1/2} t_2^{-d_2/2} \dots t_n^{-d_n/2}.$$

□

**Corollary 14:** Regions that have the same degree have the same eigenvalues.

**Proof:** Let  $r_i$  and  $r_j$  be regions with the same degree. Then by Corollary 13 since  $d(r_i) - d(r_0) = d(r_j) - d(r_0) = d(r_i)$  we know that  $m_i/m_0 = m_j/m_0 = m_j$ . Thus the corollary holds. □

Now, we have looked at an arbitrary non-abelian representation of the fundamental group of a link into  $\mathcal{G}$  and found that it must hold to certain constraints. These constraints are

$$\rho(r_i) = \begin{pmatrix} \mathbf{t}^{-d(r_i)/2} & a_i \\ 0 & \mathbf{t}^{d(r_i)/2} \end{pmatrix}$$

where  $a_i$  is some complex number. Since  $a_i$  is some complex number we can rewrite it as  $a_i = R_i \mathbf{t}^{d(r_i)/2}$ , where  $\mathbf{t}^{d(r_i)/2}$  is also a complex number.

This is exactly the construction that we used above to find that nonzero roots of the multivariable Alexander polynomial give us representations of the fundamental group into  $\mathcal{G}$ . Thus, this suggests that every non-abelian representation of the fundamental group of a link must come from solutions to the multivariable Alexander polynomial.

### 4.3 Eigenvalues and Roots of Links

I claim we can now prove that the eigenvalues of matrices in the representation of links are always derived from the roots of the multivariable Alexander Polynomial.

**Theorem 15:** The non-abelian representations of a link group,  $\pi(K)$  into  $\mathcal{G}$ , correspond to the roots of the multivariable Alexander polynomial,  $\Delta(t_i)$ .

**Proof:** Let us examine Alexander's method of constructing the Alexander polynomial.

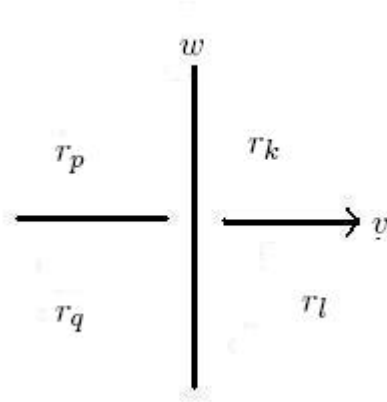


Figure 4.1: Any  $i$ th crossing in a link.

Given any crossing  $i$  (Figure 4.1) we get the  $i$ th row of the Alexander matrix

$$\begin{pmatrix} \dots & & & & \\ t_v & -1 & -t_v & 1 & 0 \\ \dots & & & & \\ \dots & & & & \end{pmatrix}$$

Similarly, looking at the fundamental link group we can once again look at the  $i$ th crossing. Every element  $\rho(r_i)$  must take the form

$$\begin{pmatrix} \mathbf{t}^{-d(r_j)/2} & R_j \mathbf{t}^{d(r_j)/2} \\ 0 & \mathbf{t}^{d(r_j)/2} \end{pmatrix}$$

with  $j$  indicating the region as discussed above. At the  $i$ th crossing we get the relation

$$r_k r_l^{-1} = r_p r_q^{-1}$$

with the eigenvalues of  $r_k$  set to  $\mathbf{x}$  and  $\mathbf{x}^{-1}$ , so that the others have eigenvalues  $\mathbf{x}t_v^{1/2}$ ,  $\mathbf{x}t_w^{1/2}$ , and  $\mathbf{x}t_v^{1/2}t_w^{1/2}$  and their inverses. This gives us

$$\begin{aligned} r_k r_l^{-1} &= \begin{pmatrix} \mathbf{x} & R_k \mathbf{x}^{-1} \\ 0 & \mathbf{x}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{x}^{-1} t_v^{-1/2} & -R_l \mathbf{x}^{-1} t_v^{-1/2} \\ 0 & \mathbf{x} t_v^{1/2} \end{pmatrix} \\ r_p r_q^{-1} &= \begin{pmatrix} \mathbf{x} t_w^{1/2} & R_p \mathbf{x}^{-1} t_w^{-1/2} \\ 0 & \mathbf{x}^{-1} t_w^{-1/2} \end{pmatrix} \begin{pmatrix} \mathbf{x}^{-1} t_v^{-1/2} t_w^{-1/2} & -R_q \mathbf{x}^{-1} t_v^{-1/2} t_w^{-1/2} \\ 0 & \mathbf{x} t_v^{1/2} t_w^{1/2} \end{pmatrix} \end{aligned}$$

yielding

$$\begin{pmatrix} t_v^{-1/2} & R_k t_v^{1/2} - R_l t_v^{-1/2} \\ 0 & t_v^{1/2} \end{pmatrix} = \begin{pmatrix} t_v^{-1/2} & R_p t_v^{1/2} - R_q t_v^{-1/2} \\ 0 & t_v^{1/2} \end{pmatrix}.$$

Demonstrating that our diagonal entries hold and so long as the upper right entries are equal this representation will work. So

$$\begin{aligned} R_k t_v^{1/2} - R_l t_v^{-1/2} &= R_p t_v^{1/2} - R_q t_v^{-1/2} \\ R_k t_v - R_l &= R_p t_v - R_q \\ R_k t_v - R_p t_v + R_q - R_l &= 0 \end{aligned}$$

Looking closely at this reveals that every crossing will result in a row of the Alexander matrix, so we now have the relation:

$$\begin{pmatrix} \dots & & & & & & & \\ \dots & t_v & -1 & -t_v & 1 & 0 & \dots & 0 \\ \dots & & & & & & & \\ \dots & & & & & & & \end{pmatrix} \begin{pmatrix} 0 \\ R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

So, to find a non-abelian representation we need to find values of the  $t_i$  and  $R_i$  such that not all  $R_i = 0$ .

This is the matrix that gives us the Alexander polynomial but we have created it from applying our representation to the fundamental group of the link. To get a nontrivial solution to this matrix equation we must be able to eliminate two columns to find the solution to the multivariable Alexander polynomial, which will give us linear independence. We know by the construction of the Dehn presentation that  $R_0$  will always equal 0 because  $r_0$  maps to the identity matrix, this eliminates one column.

The second column can be eliminated by conjugation. We are free to conjugate the representation because representations within one conjugacy class are considered equivalent. So, we want to be able to find some method

of conjugation across the representation that sends another  $R_i$  to zero, without sending all of them to zero. Conjugation within this group only changes the upper right entry of the matrix and it moves it from  $R_i \mathbf{t}^{d(r_i)/2}$  to

$$A_i = c(R_i \mathbf{t}^{d(r_i)/2} c + b(\mathbf{t}^{d(r_i)/2} - \mathbf{t}^{-d(r_i)/2})) = 0$$

when we conjugate by

$$\begin{pmatrix} c & b \\ 0 & c^{-1} \end{pmatrix}.$$

It follows, we can set  $c = 1$  so

$$b = \frac{R_i \mathbf{t}^{-d(r_i)/2}}{\mathbf{t}^{-d(r_i)/2} - \mathbf{t}^{d(r_i)/2}}.$$

But note, this is exactly the value that must be equal in all generators of an abelian representation, so there must be at least one element for which this is different. Since not all elements can have  $\mathbf{t}^{d(r_i)/2} = \pm 1$  at least one of the  $t_i$  must not equal  $\pm 1$  so an element with degree that differs from  $d(r_0)$  by one in the  $i$ th entry but does not have eigenvalues  $\pm 1$  must exist. Thus, we can conjugate by that element and some element must be different so it will not be sent to 0, and we have a nontrivial solution following conjugation and can send our second  $R_i$  to 0, getting rid of the second column.

Now to attain a nontrivial solution to this system of equations it must be linearly independent. This means that the determinant must be zero. Thus, for this relation to hold  $\Delta(t_j) = 0$  and the  $t_j$  must be roots of the multivariable Alexander polynomial. It is important to note that they must be nonzero roots, because they must have multiplicative inverses. Thus the diagonal entries must be  $\mathbf{t}^{\pm d(r_i)/2}$ , with the  $t_j$  nonzero roots of the Alexander polynomial.

This construction also gives us the constraints on the upper right entries. We know that to solve the matrix equation the vector of  $R_i$  must be in the kernel of our matrix. So, each  $R_i$  will be determined by the matrix as well and we can fully describe our representation.

Thus, we have demonstrated that if the eigenvalues correspond to the roots of the Alexander polynomial we get a representation and that the representation only occurs if the diagonal entries are roots of the Alexander polynomial.  $\square$

## 4.4 The Future

The next step in this work is to determine if the representation corresponding to the multivariable Alexander polynomial gives us more information for a link than the single variable case. One approach would be to look at how many conjugacy classes we find with each root. In the single variable case we got conjugacy classes corresponding the dimension of the null space, but it appears that the relationship is not so simple for the multivariable link case. Early research suggests that we may get a different conjugacy class for every permitted choice of eigenvalues, which would give us up to infinitely many conjugacy class per solution. This hints at a wealth of information that we may be able to begin to understand.

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