# Infinitely Many Radial Solutions to a Superlinear Dirichlet Problem

**Chee Meng Tan**

Alfonso Castro, Advisor

Jon Jacobsen, Reader

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HARVEY MUDD COLLEGE

Department of Mathematics

# Contents

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# Chapter 1

# Introduction

My thesis work started in the summer of 2005 as a three-way joint project by Professor Castro and Mr. John Kwon and myself. A paper from this joint project was written and the content now forms my thesis. In it we considered the following boundary value problem, that we dubbed a *sub-super critical* boundary value problem. The PDE is of the form

$$\begin{cases} \Delta u + g(u(x)) = 0, \ \ x \in R^N, \ \ \|x\| \leq 1 \\ u(x) = 0 \ \text{ for } \ \|x\| = 1, \end{cases} \tag{1.1}$$

where

$$g(u) = \begin{cases} u^p\,, & u \geq 0 \\ |u|^{q-1}u\,, & u < 0, \end{cases} \tag{1.2}$$

with

$$1 < p < \frac{N+2}{N-2} < q < \infty, \tag{1.3}$$

and we proved that there exists inifinitely many radial solutions to the BVP at hand. The process of doing so involves converting the ″boundary value problem″ (BVP) into an ″initial value problem″ (IVP), which yields the following

$$\begin{cases} u'' + \frac{n}{t}u' + g(u(t)) & = 0, \ \ x \in B(0,1) \\ u'(0) & = u(1) = 0, \end{cases} \tag{1.4}$$

where, and henceforth, $n = N - 1$.

## 1.1 Historical background

In 1965, Pohozaev published in Pohožaev [1965]that for a semilinear partial differential equation of the form

$$\Delta u + |u|^{\frac{p}{N-2}} u = 0, \quad x \in B(0, r) \tag{1.5}$$

has no solution for $p \geq \frac{4}{N-2}$. Furthermore (7.8) has no trivial solutions in any convex region. The results and methods employed by Pohozaev in Pohožaev [1965] is important since our stated work in the introduction section relies on the methods and results found in Pohožaev [1965]. Also, there exists the likelihood that a similar approach might also be useful in dealing with the case of the *k-Hessian*.

In 1982, Brezis and Nirenberg proved in Brézis and Nirenberg [1983] that for a boundary value problem of the type

$$\begin{cases} \Delta u + \lambda + |u|^{\frac{4}{\lambda - p}} = 0\,, & x \in B \subseteq \mathbb{R}^N \\ u(x) = 0\,, & x \in \partial B, \end{cases} \tag{1.6}$$

and found that for $N \geq 4$ for $\lambda \in (0, \lambda_1)$, (1.6) has a positive radial solution. For the case $N = 3$ there exists $\hat{\lambda}_1 \in (0, \lambda_1)$ such that (1.6) has a positive solution in $(\hat{\lambda}_1, \lambda_1)$.

Note that in their paper, Brezis and Nirenberg only considered positive solutions.

The third paper that I would like to cite has the most motivational significance in the reference list. In Castro and Kurepa [1987] Castro and Kurepa proved that when $\frac{N+1}{N-1} < g(u) < u^{\frac{N+2}{N-2}}$, (1.2) has infinitely many solutions. Since Pohožaev [1965] indicates that for $g(u) > \frac{N+2}{N-2}$, (1.2) only has a trivial solution, the natural question to ask would be the kinds of solutions that we would obtain when $g(u)$ alternates between being subcritical and supercritical. This is precisely the study we did, as mentioned in the introduction.

## 1.2 Regular Energy

In proving the existence of infinitely many solutions we used the energy that we typically find in physics, namely the total energy of a system is the sum of its potential and kinetic energy.

$$E(t,d) \equiv \frac{(u'(t,d))^2}{2} + G(u(t,d)), \tag{1.7}$$

where $G(u) = \int_0^u g(s)ds$ is our potential energy and that $\frac{(u'(t,d))^2}{2}$ is our regular kinetic energy. For future reference we note that

$$\frac{dE}{dt}(t) = -\frac{n}{t}(u'(t))^2 \leq 0. \tag{1.8}$$

## 1.3 Pohozaev Identity/Energy

Now multiplying Multiplying (1.4) by $r^{N-1}u$ and integrating on $[0,t]$, then multiplying the same equation by $r^N u'$ and integrating also on $[0,t]$ one has the following identity, known as Pohozaev's identity,

$$t^n H(t) = s^n H(s) + \int_s^t r^n (NG(u(r)) - \frac{N-2}{2} u(r) g(u(r)))\, dr, \tag{1.9}$$

The Pohozaev identity is also known as the Pohozaev Energy, which acts intuitively as a kind of energy.

*REMARK: In considering whether the solution to our ODE would oscillate all the way to $t = 1$ we needed the Pohozaev identity. For instance, a major question would be whether there was sufficient energy to allow the solution curve to rise when it has reached a minimum when the solution was in the $u < 0$ region. Or if it were to rise, would the solution start tapering off along the x-axis. Such questions appeared to be difficult in answering when just considering the normal physical energy, whereas the Pohozaev Energy tells a larger picture.

## 1.4 PDE into Radial ODE

To see that we could convert (1.1) into (1.4), let us define $u(x) = u(r)$, such that $x = (x_1, x_2, ..., x_n)$, and $u(r)$ is a radial solution to (1.1). In other words,

$u(r)$ depends only on a real variable, $r$.

From 1.1, we could rewrite the PDE into

$$u_{x_1x_1} + u_{x_2x_2} + ... + u_{x_nx_n} + g(u(x_1, x_2, ..., x_n)) = 0 \tag{1.10}$$

Assuming that u(x) is indeed radial, hence as noted above our solution to the PDE in 1.1 simply depends on one real variable, notably $r$. Hence let $u(x) = u(r)$, whereby $r = \sqrt{{x_1}^2 + {x_2}^2 + ...{x_n}^2}$. Also, note that

$$\begin{aligned} \frac{\partial r}{\partial x_k} &= \frac{1}{2}\sqrt{({x_1}^2 + {x_2}^2 + ...{x_n}^2)}.2x_k \\ &= \sqrt{({x_1}^2 + {x_2}^2 + ...{x_n}^2)}.x_k \\ &= \frac{x_k}{r} \end{aligned} \tag{1.11}$$

whereby $k \in \{1, 2, ...n\}$.

Taking the first order partial derivative with respect to the $x_k$ component, and by the chain rule we have

$$\begin{aligned} \frac{\partial}{\partial x_k}u(r) &= \frac{\partial u(r)}{r}.\frac{\partial r}{\partial x_k} \\ &= u_r.\frac{x_k}{r} \end{aligned} \tag{1.12}$$

Taking the second order derivative with respect to $x_k$,

$$\begin{aligned} \frac{\partial^2 u}{\partial {x_k}^2} &= \frac{\partial}{\partial x_k}(u_r.\frac{x_k}{r}) \\ &= u_r\frac{\partial}{\partial x_k}\frac{x_k}{r} + \frac{x_k}{r}\frac{\partial}{\partial x_k}(u_r) \\ &= u_r(\frac{x_k.(-x_k)}{r^3} + \frac{1}{r}) + \frac{x_k}{r}(u_{rr}\frac{x_k}{r}) \\ &= \frac{-{x_k}^2 u_r}{r^3} + \frac{u_r}{r} + \frac{{x_k}^2}{r^2}u_{rr} \end{aligned} \tag{1.13}$$

Now that we have an expression for each $u_{x_kx_k}$

$$
\begin{aligned}
\Delta u &= u_{x_1x_1} + u_{x_2x_2} + ... + u_{x_nx_n} \\
&= \sum_1^n \frac{-{x_k}^2 u_r}{r^3} + \frac{u_r}{r} + \frac{{x_k}^2}{r^2} u_{rr} \\
&= \sum_1^n \frac{u_r}{r^3}(r^2 - {x_k}^2) + \sum_1^n \frac{{x_k}^2}{r^2} u_{rr} \\
&= \frac{u_r}{r^3} \sum_1^n (r^2 - {x_k}^2) + u_{rr} \\
&= \frac{u_r}{r^3}(r^2 - {x_1}^2 + r^2 - {x_2}^2 + ... + r^2 - {x_n}^2) + u_{rr} \\
&= \frac{u_r}{r^3}(nr^2 - r^2) + u_{rr} \\
&= \frac{u(r)}{r^3}(n-1)r^2 + u_{rr} \\
&= (n-1)\frac{u(r)}{r} + u_{rr}
\end{aligned} \tag{1.14}
$$

Hence

$$
\begin{aligned}
\Delta(u) + g(u) &= u_{x_1x_1} + u_{x_2x_2} + ... + u_{x_nx_n} + g(u(x_1, x_2, ..., x_n)) \\
&= u_{rr} + (n-1)\frac{u(r)}{r} + g(u(r))
\end{aligned} \tag{1.15}
$$

## 1.5 Decaying Energy

We define the *energy* function by

$$
E(t,d) \equiv \frac{(u'(t,d))^2}{2} + G(u(t,d)), \tag{1.16}
$$

The derivative of $E(t,d)$ is relatively simple.Differentiatiing with respect to t, we have

$$
\begin{aligned}
\frac{dE}{dt} &= \frac{2.u'u''}{2} + G'.u' \\
&= u'u'' + g.u'
\end{aligned} \tag{1.17}
$$

Substituting the ordinary differential equation into the $\frac{dE}{dt}$, yields

$$
\begin{aligned}
\frac{dE}{dt} &= \frac{2.u'u''}{2} + G'.u' \\
&= u'u'' + g.u' \\
&= u'(\frac{-n}{t}u' - g(u)) + g.u' \\
&= \frac{-n}{t}(u')^2 - u'g(u) + g(u).u' \\
&= \frac{-n}{t}(u')^2
\end{aligned} \tag{1.18}
$$

where $G(u) = \int_0^u g(s)ds$. For future reference we note that

$$\frac{dE}{dt}(t) = -\frac{n}{t}(u'(t))^2 \leq 0. \tag{1.19}$$

## 1.6 Theorem 1

The problem (1.1) has infinitely many radial solutions.

***REMARK***:This theorem extends the results of Castro and Kurepa [1987] where it was established that if $1 < p < (N+1)/(N-1)$, or $p, q \in (1, (N+2)/(N-2))$, or $p \in (1, (N+2)/(N-2))$ and $q = (N+2)/(N-2)$, then (1.1) has infinitely many radial solutions. This result is optimal in the sense that if $p, q \in [(N+2)/(N-2), \infty)$ then $u = 0$ is the only solution to (1.1) (see Pohožaev [1965]). For related results for quasilinear equations the reader is referred to García-Huidobro et al. [1997]. For early studies on the critical case, $p = q = (N+2)/(N-2)$, see Atkinson et al. [1988], Brézis and Nirenberg [1983], Castro and Kurepa [1987], Castro and Kurepa [1989], Castro and Kurepa [1994], and Cerami et al. [1986]. In Benguria et al. [2000] the reader finds an complete classification of the radial solutions to (1.1) for $1 < p = q < (N+2)/(N-2)$. For a recent survey of radial solutions for elliptic boundary value problems, which includes the case where the Laplacian operator is replaced by the more general *k-Hessian* operator, see Jacobsen and Schmitt [2004].

## 1.7 Theorem 2

There exists $D > 0$ such that if $d \geq D$ then

$$t^{N-1}(tE(t) + \frac{N-2}{2}u(t)u'(t)) \geq cd^{\xi} \text{ for all } t \geq \sqrt{N}d^{(1-p)/2}, \tag{1.20}$$

where $\xi = \frac{N+2-p(N-2)}{2}$. Also $u(t) \geq d/2$ for $t \in [0, \sqrt{N} d^{(1-p)/2}]$.

## 1.8 Lemma 1

As a consequence of Theorem 2 we see that there exists a continuous function $\theta : [0,1] \times [D, \infty) \to \mathbb{R}$ such that

$$u(t,d) = \rho(t,d)\cos(\theta(t,d)) \text{ and } u'(t,d) = \rho(t,d)\sin(\theta(t,d)). \tag{1.21}$$

## 1.9 Theorem 3

$$\lim_{d\to\infty} \theta(1,d) = +\infty. \tag{1.22}$$

# Chapter 2

# First Zero

## 2.1 First Zero

Let $d > 0$ and $t_0 > 0$ be such that $u(t_0) = d/2$, and $u(t) > d/2$ for $t \in (0, t_0)$. The following derivation is based on results obtained from [5].

$$-u'(t) = t^{-n} \int_0^t s^n g(u(s)) ds \tag{2.1}$$

To obtain the above, simply multiply 1.4 by $-t^n$, which yields

$$\begin{aligned} -t^n u'' - n.(t^{n-1})u' &= t^n g(u) \\ -(t^n u')' &= t^n g(u) \end{aligned} \tag{2.2}$$

$$t^n u' = -\int_0^t t^n g(u)\, ds \tag{2.3}$$

Now, we proceed to integrate the above expression from 0 to $r$, and since $u'(0)$=0, we obtain (2.1)

**REMARK:**

Having this equation is a powerful tool, since we MAY estimate how rapidly the graph falls from $t = 0$. We know that regular energy decreases as a function of time from 1.19, which means that the solution curve as a function of time cannot be increasing. So the curve of the slope has to become negative. However, the rapidity of the decreasing slope is of question, and with (2.1), we are able to estimate the gradient of the curve at say $t_0$, whereby $u(t_0) = \frac{d}{2}$, and also estimate what $t_0$ is. Let us see that

$$\sqrt{N} d^{(1-p)/2} \leq t_0 \leq \sqrt{2^p N} d^{(1-p)/2}. \tag{2.4}$$

To show the above, simply look at the interval of $[0, t_0]$. Knowing that since $u$ is decreasing with time, this implies that $u < d$. Hence from (2.1),

$$\begin{aligned} u'(t) &\leq t^{-n} \int_0^t s^n g(u) ds \\ &= t^{-n} d^p \frac{t^{n+1}}{n+1} \end{aligned}$$

Integrating $u'(t)$ from 0 to $t_0$, yields

$$\begin{aligned} -\int_0^{t_0} u'(t) dt &\leq \int_0^{t_0} \frac{d^p}{n+1} t dt \\ -u(t_0) + u(0) &\leq \frac{d^p}{n+1} \frac{(t_0)^2}{2} \\ \frac{-d}{2} + d &\leq \frac{d^p}{n+1} \frac{(t_0)^2}{2} \end{aligned}$$

Hence

$$(t_0)^2 \geq \frac{d^{1-p}}{n+1} \tag{2.5}$$

Similarly since $u(t) \geq \frac{d}{2}$ for $t \in [0, t_0]$, and repeating the same process as above in finding the lower-bound of $t_0$, we will find that an upper-bound of $t_0 \leq \sqrt{2^p N} d^{(1-p)/2}$

Now multiplying Multiplying (1.4) by $r^{N-1} u$ and integrating on $[0, t]$, then multiplying the same equation by $r^N u'$ and integrating also on $[0, t]$ one has the following identity, known as Pohozaev's identity,

$$t^n H(t) = s^n H(s) + \int_s^t r^n \left( NG(u(r)) - \frac{N-2}{2} u(r) g(u(r)) \right) dr, \tag{2.6}$$

where $H(x) \equiv xE(x) + \frac{N-2}{2} u'(x) u(x)$. In particular, taking $s = 0$ and $t = t_0$ we have

$$t_0^n H(t_0) \geq \frac{t_0^N \gamma d^{p+1}}{2^{p+1} N} \geq \frac{N^{N/2} \gamma}{2^{p+1} N} d^{\xi} \equiv c_1 d^{\xi}, \tag{2.7}$$

where $\gamma = N/(p+1) - (N-2)/2$ and $\xi$ is as in (1.20). Thus if $u(s) \geq 0$ for all $s \in [0, t]$ we have

$$t^N (u'(t))^2 = -(N-2) u \cdot t^n u' - 2t^N \frac{u^{p+1}}{p+1} + 2 \int_0^t \gamma s^n u^{p+1} \, ds. \tag{2.8}$$

Also

$$
\begin{aligned}
\left(\frac{-tu'}{u}\right)' &= \frac{(-tu''-u')u+t(u')^2}{u^2} \\
&= \frac{-t(-\frac{n}{t}u'-u^p)u-uu'+t(u')^2}{u^2} \\
&= \frac{(\overbrace{n-1}^{N-2})uu'+tu^{p+1}+t(u')^2}{u^2} \\
&= \frac{2t^{-n}\int_0^t s^n\gamma u^{p+1}\,ds-2\frac{tu^{p+1}}{p+1}+tu^{p+1}}{u^2} \\
&= \frac{2t^{-n}\int_0^t s^n\gamma u^{p+1}\,ds+t\left(\frac{p-1}{p+1}\right)u^{p+1}}{u^2} \\
&\geq \frac{2\gamma}{t}\left(\frac{-tu'(t)}{u(t)}\right),
\end{aligned}
\tag{2.9}
$$

provided $u(s) > 0$ for $s \in (0,t)$. Integrating (2.9) on $[t_0,t)$ we have

$$
\ln\left(\frac{\frac{-tu'(t)}{u(t)}}{\frac{-t_0u'(t_0)}{u(t_0)}}\right) \geq \ln(\frac{t}{t_0})^\gamma. \tag{2.10}
$$

Letting $\Gamma = \frac{-t_0u'(t_0)}{u(t_0)}$ we conclude

$$
\frac{-tu'(t)}{u(t)} \geq \Gamma\left(\frac{t}{t_0}\right)^\gamma. \tag{2.11}
$$

For future reference we note that

$$
\Gamma \geq 2^{1-p}, \tag{2.12}
$$

where we have used (2.4), and $-u'(t_0) \geq t_0d^p/(2^pN)$ (see (2.1)). Integrating again in $[t_0,t]$ yields

$$
\ln\left(\frac{u(t_0)}{u(t)}\right) \geq \frac{\Gamma}{\gamma t_0^\gamma}[t^\gamma - t_0^\gamma]. \tag{2.13}
$$

Assuming that $u(t) \geq 0$ for all $t \in [t_0, t_0\ln^{1/\gamma}(d) \equiv T]$ we have

$$
u(T) \leq u(t_0)(ed^{-1})^{\Gamma/\gamma} = \frac{e^{\Gamma/\gamma}}{2}d^{1-\Gamma/\gamma}. \tag{2.14}
$$

Now we estimate $E$ for $t \geq t_0$ with $u(s) \geq 0$ for $s \in (t_0, t]$. Since $E'(t) \geq -2nE(t)/t$,

$$E(t) \geq E(s)(s/t)^{2n} \text{ for any } \ 0 \leq s \leq t \leq 1. \tag{2.15}$$

Thus

$$\begin{aligned}\frac{(u'(T))^2}{2} &\geq E(t_0)(\frac{t_0}{T})^{2n} - \frac{u^{p+1}(T)}{p+1} \\ &\geq \frac{d^{p+1}}{(p+1)2^{p+1}\ln^{2n/\gamma}(d)} - \frac{1}{p+1}\left(\frac{e^{\Gamma/\gamma}}{2}d^{1-\Gamma/\gamma}\right)^{p+1} \\ &\geq \frac{d^{p+1}}{(p+1)2^{p+2}\ln^{2n/\gamma}(d)},\end{aligned} \tag{2.16}$$

for $d$ sufficiently large. Assuming again that $u(T) > 0$, for any $t \in [T, 2T]$ such that $u(s) > 0$ for any $s \in [T, t)$ arguing as in (2.16) we have

$$\begin{aligned}\frac{(u'(t))^2}{2} &\geq E(T)(\frac{T}{t})^{2n} - \frac{u^{p+1}(T)}{p+1} \\ &\geq \frac{d^{p+1}}{(p+1)2^{p+1+2n}\ln^{2n/\gamma}(d)} - \frac{1}{p+1}\left(\frac{e^{\Gamma/\gamma}}{2}d^{1-\Gamma/\gamma}\right)^{p+1} \\ &\geq \frac{d^{p+1}}{(p+1)2^{p+2+2n}\ln^{2n/\gamma}(d)},\end{aligned} \tag{2.17}$$

for $d$ large. Integrating on $[T, t]$ we have

$$\begin{aligned}0 \leq u(t) &= u(T) + \int_T^t u'(s)ds \\ &\leq \frac{e^{\Gamma/\gamma}}{2}d^{1-\Gamma/\gamma} - (t-T)\frac{\sqrt{2}d^{(p+1)/2}}{2^{1+n+(p/2)}\ln^{n/\gamma}(d)\sqrt{p+1}}.\end{aligned} \tag{2.18}$$

Hence $u$ has a zero in $[d^{(1-p)/2}, T + e^{\Gamma/\gamma}d^{(1-p)/2-\Gamma/\gamma}2^{n+(p/2)}\ln^{n/\gamma}(d)\sqrt{p+1}]$. We summarize the above in the following lemma.

**Lemma 2.1** *For $d > 0$ sufficiently large, there exists*

$$t_1 \in (\sqrt{N}d^{(1-p)/2}, 2d^{(1-p)/2}\ln^{1/\gamma}(d)) \tag{2.19}$$

*such that $u(t_1) = 0$, $u(s) > 0$ for $s \in [0, t_1)$, and*

$$\frac{d^{p+1}}{(p+1)2^{p+2+2n}\ln^{2n}(d)} \leq E(t_1) \leq \frac{d^{p+1}}{p+1} \tag{2.20}$$

# Chapter 3

# First Local Minimum

Let $t \in (t_1, t_1 + (1/2)(2/(q+1))^{q/(q+1)}|u'(t_1)|^{(1-q)/(1+q)} \equiv t_1 + \tau)$. From (2.19) we see that $t_1/t \geq 1 + md^{(p-q)/(p+q)}$ with $m$ independent of $d > 0$ for $d$ large. Hence for $d > 0$ large

$$\begin{aligned} u'(t) &= t^{-n}[t_1^n u'(t_1) - \int_{t_1}^{t} s^n |u(s)|^{q-1} u(s) ds] \\ &\leq (.9)^n u'(t_1) + (t - t_1) \left(\frac{q+1}{2}\right)^{q/(q+1)} |u'(t_1)|^{(2q)/(q+1)} \\ &\leq (.4)^n u'(t_1), \end{aligned} \tag{3.1}$$

where we have used that, since $E' \leq 0$, $|u|^{q+1}(t) \leq (q+1)(u'(t_1))^2/2$ for $t \geq t_1$ with $u(t) \leq 0$. This, and (3.3) yield

$$\begin{aligned} u(t_1 + \tau) &\leq (.4)^n u'(t_1)\tau \\ &\leq -(.4)^n |u'(t_1)|^{2/(1+q)}. \end{aligned} \tag{3.2}$$

Now for $t \geq t_1 + \tau$ with $u(s) \leq -(.4)^n |u'(t_1)|^{2/(1+q)}$ for all $s \in (t_1 + \tau, t)$ we have

$$\begin{aligned} u'(t) &= t^{-n}[t_1^n u'(t_1) - \int_{t_1}^{t} s^n |u(s)|^{q-1} u(s) ds] \\ &\geq u'(t_1) + t^{-n}(.4)^{qn} |u'(t_1)|^{2q/(1+q)} \int_{t_1+\tau}^{t} s^n ds \\ &\geq -u'(t_1)[-1 + t^{-n}(.4)^{qn} |u'(t_1)|^{(q-1)/(1+q)} \frac{t^N - (t_1+\tau)^N}{N}] \\ &\geq -u'(t_1)[-1 + \frac{(.4)^{qn}}{N} |u'(t_1)|^{(q-1)/(1+q)} (t - (t_1 + \tau))]. \end{aligned} \tag{3.3}$$

This and (3.3) imply the following lemma.

## 3.1 Lemma 2

**Lemma 3.1** *There exists* $\tau_1 \in (t_1, t_1 + \{(1/2)(2/(q+1))^{q/(q+1)} + \frac{N}{(.4)^{qn}}\}|u'(t_1)|^{(1-p)/(1+q)}]$ *such that* $u'(\tau_1) = 0$.

# Chapter 4

# Second Zero

Let $\tau_0 > \tau_1$ be such that $u(s) \leq .5u(\tau_1)$ for all $s \in [\tau_1, \tau_0]$. Imitating the arguments leading to (2.4) we see that

$$\tau_1 + |u(\tau_1)|^{(1-q)/2} \leq \tau_0 \leq \tau_1 + \sqrt{2^q N}|u(\tau_1)|^{(1-q)/2}. \tag{4.1}$$

Hence

$$u'(\tau_0) \geq \frac{|u(\tau_1)|^{(1+q)/2}}{2^q N}, \tag{4.2}$$

and

$$\tau_0^n \geq .9s^n \quad \text{for any} \quad s \in (\tau_0, \tau_0 + 2^{q+2}N|u(\tau_1)|^{(1-q)/2}], \tag{4.3}$$

for $d > 0$ sufficiently large. Thus, if for some $r \in [\tau_0 + 2^{q+1}N|u(\tau_1)|^{(1-q)/2}, \tau_0 + 2^{q+2}N|u(\tau_1)|^{(1-q)/2}]$, $u(s) \leq 0$ for all $s \in [\tau_0, r]$ we have

$$u'(s) \geq .9u'(\tau_0) \quad \text{for all} \quad x \in [\tau_0, r]. \tag{4.4}$$

This and and the definition of $r$ give

$$\begin{aligned} 0 \geq u(r) \geq \frac{u(\tau_1)}{2} + (r - \tau_0).9u'(r) \\ \frac{u(\tau_1)}{2} + 2^{q+1}N|u(\tau_1)|^{(1-q)/2}.9\frac{|u(\tau_1)|^{(1+q)/2}}{2^q N}, \end{aligned} \tag{4.5}$$

which is a contradiction. Thus we have proven:

## 4.1 Lemma 3

**Lemma 4.1** *There exist* $t_2 \in [t_1, t_1 + k|u'(t_1)|^{(1-q)/(1+q)}]$ *such that* $u(t_2) = 0$ *and* $u(s) < 0$ *in* $(t_1, t_2)$.

For $t > \tau_0$with $u(r) \leq 0$ for all $r \in [\tau_0, t]$ we have

$$\begin{aligned} u'(t) &= t^{-n}[t_1^n u'(t_1) - \int_{t_1}^{t} s^n |u(s)|^{q-1} u(s) ds] \\ &\leq (.9)^n u'(t_1) + (t - t_1) \left( \frac{q+1}{2} \right)^{q/(q+1)} |u'(t_1)|^{(2q)/(q+1)} \\ &\leq (.4)^n u'(t_1), \end{aligned} \tag{4.6}$$

# Chapter 5

# First Positive Minimum

## 5.1 Second Minimum

Let $t > t_2$ be such that $u'(s) > 0$ on $[t_2, t]$. Thus $u'' \leq 0$ in $[t_2, t]$. Hence $u(s) \leq u'(t_2)(s - t_2)$ for all $s \in [t_2, s]$, $s \in [t_2, t]$. integrating (1.4) on $[t_2, s]$ we have

$$\begin{aligned} s^n u'(s) &= t_2^n u'(t_2) - \int_{t_2}^{s} r^n |u(r)|^{p-1} u(r) dr \\ &\geq t_2^n u'(t_2) - s^n \frac{|u'(t_2)|^p (s - t_2)^{p+1}}{p+1} \\ &\geq u'(t_2) \left( t_2^n - \frac{s^n}{p+1} \right), \end{aligned} \tag{5.1}$$

for $s \leq t_2 + u'(t_2)^{(1-p)/(1+p)}$. Since $t_2^N |u'(t_2)|^2 \geq 2c_1 d^{\xi}$ (see (2.7)) and $(u'(t_2))^2 \leq 2d^{p+1}/(p+1)$, we have

$$t_2^N \geq 2c_1 \left( \frac{p+1}{2} \right)^{\xi/(p+1)} |u'(t_2)|^{N(1-p)/(1+p)}. \tag{5.2}$$

Now for

$$s \in [t_2, \min\{2^{1/n}, 1 + (2c_1)^{-1/N} \left( \frac{2}{p+1} \right)^{\frac{\xi}{N(p+1)}} \} t_2 \equiv \alpha t_2],$$

from (5.1) and (5.2 ) we have

$$u'(s) \geq u'(t_2) \left( \frac{t_2^n}{s^n} - \frac{1}{p+1} \right) \geq u'(t_2) \frac{p-1}{p+1}. \tag{5.3}$$

Integration on $[t_2, \alpha t_2]$ yields

$$u(\alpha t_2) \geq \frac{p-1}{p+1} \alpha t_2 u'(t_2). \tag{5.4}$$

Therefore

$$\begin{aligned} t^n u'(t) &\leq t_2^n u'(t_2) - \int_{\alpha t_2}^{t} r^n |u(r)|^{p-1} u(r) dr \\ &\leq t_2^n u'(t_2) - t_2^n (t - \alpha t_2) \left( \frac{p-1}{p+1} \alpha t_2 u'(t_2) \right)^p . \end{aligned} \tag{5.5}$$

This and (5.2) imply

$$\begin{aligned} t - \alpha t_2 &\leq \left( \frac{p-1}{p+1} \alpha \right)^{-p} t_2^{-p} |u'(t_2)|^{1-p} \\ &\leq c_3 |u'(t_2)|^{(1-p)/(p+1)}. \end{aligned} \tag{5.6}$$

This concludes the following.

**Lemma 5.1** *There exists* $\tau_2 \in [t_2, \alpha t_2 + c_3 |u'(t_2)|^{(1-p)/(p+1)}]$ *such that* $u'(\tau_2) = 0$ *and* $u'(s) > 0$ *on* $[t_2, \tau_2)$.

# Chapter 6

# Energy Analysis

## 6.1 Energy on $[t_0, \tau_2]$

Let us see that

$$\int_{t_0}^{t_1} t^n \gamma u^{p+1}(t)dt \geq \int_{t_1}^{t_2} t^n \gamma_1 |u(t)^{q+1}|dt, \tag{6.1}$$

where $\gamma_1 = ((q+1)(N-2) - 2N)/(2(q+1))$. Let $\hat{t}_0 \in [t_0, t_1]$ be such that $u(\hat{t}_0) = d/4$. Then, for $t \in [t_0, \hat{t}_0]$, we have

$$-u'(t) = t^{-n} \int_0^t s^n u^p(s)\, ds \leq \frac{td^p}{N}. \tag{6.2}$$

Integrating on $[t_0, \hat{t}_0]$ we have $(d/4) \leq (\hat{t}_0^2 - t_0^2)d^p/(2N)$. This and (2.4) yield

$$\hat{t}_0 \geq \sqrt{\frac{Nd^{1-p}}{2} + t_0^2} t_0 \sqrt{1 + \frac{Nd^{1-p}}{2t_0^2}} \geq t_0 \sqrt{1 + \frac{1}{2^{p+1}}}. \tag{6.3}$$

This and (2.4) give

$$\begin{aligned}
\int_{t_0}^{t_1} t^n \gamma u^{p+1}(t)dt &\geq \int_{t_0}^{\hat{t}_0} t^n \gamma u^{p+1}(t)dt \\
&\geq \gamma (d/4)^{p+1} \frac{\hat{t}_0^N - t_0^N}{N} \\
&\geq \frac{\gamma}{4^{p+1}N} t_0^N \left( \left(1 + \frac{1}{2^{p+1}}\right)^{N/2} - 1 \right) d^{p+1} \\
&\geq \frac{\gamma}{4^{p+1}N} \left( \left(1 + \frac{1}{2^{p+1}}\right)^{N/2} - 1 \right) N^{N/2} d^{\xi}.
\end{aligned} \tag{6.4}$$

Using (1.19), we have $|u(t)|^{q+1} \le (q+1)d^{p+1}/(p+1)$. This, Lemma 4.1,

$$\begin{aligned}
\int_{t_1}^{t_2} t^n |u|^{q+1}(t)dt &\le \left(\frac{q+1}{p+1}d^{p+1}\right)\frac{t_2^N - t_1^N}{N} \\
&\le \left(\frac{q+1}{p+1}d^{p+1}\right)\frac{(t_1 + k|u'(t_1)|^{(1-q)/(1+q)})^N - t_1^N}{N} \\
&\le \left(\frac{q+1}{p+1}d^{p+1}\right) t_1^n \frac{(2^N-1)k|u'(t_1)|^{(1-q)/(1+q)}}{N}.
\end{aligned} \tag{6.5}$$

Also from (7.8) and (2.7) we have $t_1 H(t_1) \ge c_1 d^{\xi}$. Replacing this in (6.4) and using Lemma 2.1 we have

$$\begin{aligned}
\int_{t_1}^{t_2} t^n |u|^{q+1}(t)dt &\le \left(\frac{q+1}{p+1}d^{p+1}\right)\frac{(2^N-1)k}{N}\, t_1^n \left(d^{\xi/2} t_1^{-N/2}\right)^{\frac{1-q}{1+q}} \\
&= \left(\frac{q+1}{p+1}\right)\frac{(2^N-1)k}{N}\, d^{p+1+(\xi/2)\left(\frac{1-q}{1+q}\right)} t_1^{N-1-\frac{N}{2}\frac{1-q}{1+q}} \\
&\le \left(\frac{q+1}{p+1}\right)\frac{(2^N-1)k}{N} \ln^{M/\gamma}(d)\, d^{\eta},
\end{aligned} \tag{6.6}$$

where

$$\begin{aligned}
\eta &= p+1+\frac{\xi(1-q)}{1+q} + \frac{(1-p)(2(N-1)(1+q)-N(1-q))}{2(1+q)}, \\
M &= (2(N-1)(1+q)-N(1-q))/(2(1+q)),
\end{aligned} \tag{6.7}$$

and the factor $2^N - 1$ comes from the fact that $k|u'(t_1)|^{(1-q)/(1+q)} < t_1$ for $d > 0$ large.

An elementary calculation shows that $\xi > \eta$. Thus from (6.4) and (6.6), (6.1) follows.

Let now $t \in [t_1, \tau_2]$. Since

$$\begin{aligned}
t^n H(t) &= t_0^n H(t_0) + \int_{t_0}^{t} s^n \left(NG(u(s)) - \frac{N-2}{2}u(s)g(u(s))\right)ds \\
&\ge t_0^n H(t_0) + \int_{t_0}^{t_2} s^n \left(NG(u(s)) - \frac{N-2}{2}u(s)g(u(s))\right)ds \\
&\ge t_0^n H(t_0) \\
&\ge c_1 d^{\xi}.
\end{aligned} \tag{6.8}$$

# Chapter 7

# Proof of Main Theorem

## 7.1 The Core Idea

Arguing as in lemmas 2.1 and 4.1, we see that for $d > 0$ sufficiently large there exist numbers

$$t_3 < \cdots < t_k \leq 1 \tag{7.1}$$

such that

$$u(t) < 0 \text{ in } (t_{2i-1}, t_{2i}), \text{ and } u(t) > 0 \text{ in } (t_{2i}, t_{2i+1}), \; i = 1, \ldots \min\left\{\frac{k}{2}, \frac{k+1}{2}\right\}. \tag{7.2}$$

Imitating the arguments leading to (6.1) one sees that

$$\int_{t_{2i}}^{t_{2i+1}} t^n \gamma u^{p+1}(t)dt \geq \int_{t_{2i+1}}^{t_{2i+2}} t^n \gamma_1 |u(t)^{q+1}| dt. \tag{7.3}$$

This in turn (see (6.8)) leads to

$$t^n H(t) \geq c_1 d^{\xi} \;\text{ for all }\; t \in [t_0, 1]. \tag{7.4}$$

This, together with lemma 2.1, proves Theorem **??**. From (7.4) we see that

$$\rho^2(t) \equiv u^2(t) + (u'(t))^2 \to \infty \;\text{ as } d \to +\infty, \tag{7.5}$$

uniformly for $t \in [0, 1]$. Therefore, there exists a continuous *argument* function $\theta(t, d) \equiv \theta(t)$ such that

$$u(t) = \rho(t)\cos(\theta(t)) \;\text{ and }\; u'(t) = -\rho(t)\sin(\theta(t)). \tag{7.6}$$

From this we see that $\theta'(t) = \{((n/t)u'(t) + g(u(t)))u(t) + (u'(t))^2\}/\rho^2(t)$. Thus $\theta'(t) > 0$ for $\theta(t) = j\pi/2$ with $j = 1, \ldots$, which implies that if

$\theta(t) = j\pi/2$ then $\theta(s) > j\pi/2$ for all $s \in (t, 1]$.

Imitating the arguments of lemma 2.1 and 4.1 we see that $t_{2i} - t_{2(i-1)} \leq c_3 ln^{1/\gamma}(d) d^{(1-p)/2}$. Thus (see (7.1)) $k \geq c_4 ln^{-1/\gamma}(d) d^{(p-1)/2}$, which implies that

$$\lim_{d \to +\infty} \theta(1, d) = +\infty. \tag{7.7}$$

Hence we have proven Theorem (3). By the continuity of $\theta$ we see that there exists a sequence $d_1 < \ldots < d_j < \ldots \to \infty$ such that $\theta(1, d_j) = j\pi + (\pi/2)$. Hence $u(t, d_j)$ is a solution to (1.1) having exactly $j$ zeroes in $(0, 1)$, which proves Theorem 1.6.

***IMPORTANT REMARK** to 7.5 and 7.6:
Although the Pohozaev Energy in my research is

$$t^n H(t) = s^n H(s) + \int_s^t (r^n (NG(u(r)) - \frac{N-2}{2} u(r) g(u(r))) dr, \tag{7.8}$$

Based on my current research with Professor Alfonso Castro we generalized the Pohozaev Energy as the following

$$\begin{aligned} H(r) &= r^{n+\beta} \frac{u^2}{2} + r^{N+\alpha} F(u) + \frac{N+\beta-2}{2} r^{n+\beta-1} u u' \\ &= \int_0^r s^{n+\alpha-1} (N+\alpha) F(u) - \frac{N+\beta-2}{2} u f(u)) ds \end{aligned} \tag{7.9}$$

and with the assumption that f is to be subcritical for $u > 0$ and $(N + \alpha)F(u) - \frac{N+\beta-2}{2} u f(u)$ is bounded from below, we conclude that $H(r) \geq k_1 d^i - k_2$ for $r \geq r_0$ whereby $i > 0$.

This implies that $\rho^2 = u^2(r) + (u')^2 \geq M > 0$ for any $M \in \mathbb{R}$. The question is why is this necessarily true?

Assuming that $u^2(r) + (u'(r))^2 \leq M$ for some $M > 0$, for any value of $u(0) = d$. Then either $u^2$ or $(u')^2$ must be less than $\frac{M}{2}$. Furthermore, both $u^2$ and $(u')^2$ must both be less than M at any given value of $r \in [0, 1]$ This would imply that the Pohozaev Energy identity listed above would be bounded, since,

**i)** $r \in [0,1]$ this r is bounded.

**ii)** $u$ and $(u')^2$ is bounded, and since the non-linearity $F(u)$ here is $u^p$, with $p = \frac{N+2}{N-2}$, all the terms in the first Pohozaev inequality is bounded for any value of $u(0) = d$. This is a contradiction, hence for arbitrarily large $d$'s, $u^2(r) + (u'(r))^2$ is bounded away from $u = 0$ and $u' = 0$.

Now let us see that there exists a $\theta : [0,1] \to [0,\infty]$ that depends continuously on the initial condition $u(0) = d$, with

$$u(r) = \rho cos(\theta(r))$$

and

$$u'(r) = -\rho sin(\theta(r))$$

The way we define $u'(r)$ as having a negative sign in front of the expression simply means that we are rotating clockwise as opposed to the usual anticlockwise convention that is commonly used in measuring the angle.

Let us define $\theta(0) = 0$, so that we start off our angle in the phase-plane on the $x$-axis.

Let $\epsilon > 0$ be such that

$$u(r) > 0 \quad \text{for} \quad r \in [0,\epsilon]$$

on [0,$\epsilon$), we let $\theta(r) = arcos(\frac{u(r)}{\rho(r)})$

Hence in [0,$\epsilon$), $-u'(r) = \rho(r)sin(\theta(r))$. Let $\delta_1$=sup($\delta \in [0,1]$). Since arcos($\frac{u(r)}{\rho(r)}$) is defined, it makes sense to calculate $\theta'(r)$.

Suppose that without lost of generality that $u'(r) \neq 0$, then we know that since $u'(r)$ is defined as the $y$-axis of the phase-plane and $u(r)$ represents the $x$-axis on the phase plane,

$$\frac{u(r)}{u'(r)} = \frac{cos(\theta(r)}{sin(\theta(r))}$$
$$= cotan(\theta)$$

Taking the derivative on both sides yields

$$\begin{aligned}(\frac{u(r)}{u'(r)})' &= -(cotan(\theta))' \\ &= cosec(\theta(r)) \\ &= \frac{\theta'(r)}{(-u'/\rho)^2}\end{aligned}$$

Hence $\theta(r)$ becomes

$$\theta'(r) = \frac{-(u')^2 + uu''}{(\rho)^2} \tag{7.10}$$

Now let us assume that $u(r) \neq 0$, and since

$$tan(\theta(r)) = \frac{-u'(r)}{u(r)}$$

this implies that

$$\theta(r) = arctan(\frac{-u'(r)}{u(r)})$$

Since $(arctan(x))' = \frac{1}{1+x^2}$, this means that

$$\begin{aligned}\theta(r) &= \frac{1}{1 + \frac{(u')^2}{u^2}} . \frac{u''u - (-u')u'}{u^2(r)} \\ &= \frac{(u')^2 - uu''}{u^2 + (u')^2}\end{aligned} \tag{7.11}$$

Since (7.10) and (7.11) yield the same identity for $\theta'(r)$ this implies that

$$\theta(r) = \frac{(u')^2 - uu''}{u^2 + (u')^2}$$

universally on the phase-plane

## 7.2 Phase Plane Analysis

In the paper written by Professor Alfonso Castro and Professor Alexanra Kurepa, the PDE that was considered is of the form

$$\Delta u + g(u) = p(t) \tag{7.12}$$

whereby $g(u)$ is superlinear, which means that $\frac{g(u)}{u} \to \infty$ as $u \to \infty$ and that $p(t)$ is bounded and continuous. In our joint paper with Mr. John Kwon and Professor Alfonso Castro, we simply set $p(t) = 0$. In doing so, what it means is that Theorem (1) can be proven using the phase plane analysis technique employed in the Castro & Kurepa paper, by simply setting $p(t) = 0$.

Since u depends continuously on parameters $(t, d)$, we should write $u = u(t, d)$. Hence

$$\begin{aligned} u(t,d) &= r(t,d)cos(\theta(t,d)) \\ u'(t,d) &= -r(t,d)sin(\theta(t,d)) \\ \theta(0,d) &= 0 \end{aligned} \tag{7.13}$$

The reason for $\theta(0, d) = 0$ stems from the fact that the phase plane analysis has $u(r)$ as the $x - axis$ and $u'(r)$ as the $y - axis$. When $t = 0$ the $u - t$ graph starts off at $u(0) = d$ with $u'(0) = 0$. Which means that we start off on the $x$-axis on the phase plane.

In order to prove theorem (1), we need to show that

$$\lim_{d\to\infty} \theta(1, d) = +\infty. \tag{7.14}$$

In order to do so, we show that given any positive integer, J, there exists $d_0$ such that if $d \geq d_0$, then $\theta(T, d) > J\pi$

If $x_0 > 0$ and $m(x_0) := min\{g(x)/x : |x| > x_0\}$, then by the superlinear property of $g(u)$, that is

$$\lim_{u\to\infty} \frac{g(u)}{u} = +\infty. \tag{7.15}$$

we have

$$\lim_{x_0\to\infty} m(x_0) = +\infty. \tag{7.16}$$

Now let us define

$$\delta \in (0, min\{\pi/4, T/64n\}) \tag{7.17}$$

and $r_0$ be such that

$$8\|p\| < r_0 \tag{7.18}$$

$$16\delta + 2\pi/\omega(r_0,\delta) \leq 3T/4J \tag{7.19}$$

$$\omega(r_0,\delta) > (4n/T) + (2\|p\|_\infty)/r_0 \tag{7.20}$$

where $\omega(r_0,\delta) = m(r_0 cos\delta)sin^2(\delta)$

Since $r(t,d) \to \infty$ as $E(t,d) \to \infty$ we see that by continuity of $E$ with respect to $d$, that given $d > d_0$, $E > E_0$ for some $E_0$ for all $t \in [1,d]$. Hence $\theta(t,d)$ is defined on all $t \in [1,d]$

Suppose that $t \geq (T/4)$ and $\theta(t,d)$ belongs to an interval of the form $[j\pi/2 - \delta, j\pi/2 + \delta]$ where j is a non-negative odd integer. From

$$\theta'(t,d) = sin^2(t,d) + \frac{(g(u(t,d)) + \frac{n}{t}u'(t,d) - p(t))cos\theta(t,d)}{r(t,d)} \tag{7.21}$$

we have that

$$\begin{aligned}\theta'(t,d) &\geq sin^2\theta(t,d) + \frac{g(u(t,d))u(t,d)}{r^2(t,d)} - \frac{4n|u(t,d)u'(t,d)|}{Tr^2(t,d)} - \frac{\|p\|_\infty}{r_0}\\ &\geq cos^2\delta - \frac{4n.\tan\delta}{T} - \frac{\|p\|_\infty}{r_0}\\ &\geq \frac{1}{4}\end{aligned} \tag{7.22}$$

Using the (7.17), (7.18) and (7.19).

On the other hand, if $\theta(t,d)$ belongs to an interval of the form $[\frac{j\pi}{2} + \delta, \frac{(j+2)\pi}{2} - \delta]$. We have

$$\begin{aligned}\theta'(t,d) &\geq \frac{g(u)sin^2\delta}{u} - \frac{2n}{T} - \frac{\|p\|_\infty}{r_0}\\ &\geq \omega(r_0,\delta)/2\\ &> 0\end{aligned} \tag{7.23}$$

Since $\theta'(t) > 0$ for all $t$ this implies that the curve of $uvsu'$ curve always spirals in a clockwise direction . As $d \to \infty$ the spiral not only becomes larger, but the number of times that it intersects the y-axis also increases, thus giving rise to the infinite number of radial solutions that we seek.

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