

A Tiling Approach to Chebyshev Polynomials

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Abstract

We present a combinatorial interpretation of Chebyshev polynomials. The n th Chebyshev polynomial of the first kind, $T_n(x)$, counts the sum of all weights of n -tilings using light and dark squares of weight x and dominoes of weight -1 , and the first tile, if a square must be light. If we relax the condition that the first square must be light, the sum of all weights is the n th Chebyshev polynomial of the second kind, $U_n(x)$. In this paper we prove many of the beautiful Chebyshev identities using the tiling interpretation.

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Chapter 1

Introduction

1.1 Background

Chebyshev polynomials arise in a variety of continuous settings. They are a sequence of orthogonal polynomials appearing in approximation theory, numerical integration, and differential equations (see Rivlin (1990) for examples). In this paper we approach them instead as discrete objects, counting the sum of the weights of weighted tilings. Our combinatorial approach will allow us to prove identities holding for these continuous functions using discrete arguments.

Chebyshev polynomials can be defined in a number of ways. Perhaps the most common way is in terms of the following surprising formula. The n th Chebyshev polynomial of the first kind satisfies, for $0 \leq \theta \leq \pi$,

$$\cos(n\theta) = T_n(\cos(\theta)). \quad (1.1)$$

Meanwhile, the n th Chebyshev polynomial of the second kind satisfies a related formula: for $0 < \theta < \pi$,

$$\frac{\sin(n+1)\theta}{\sin(\theta)} = U_n(\cos \theta). \quad (1.2)$$

More important to our study is the recurrence that the Chebyshev polynomials follow. An alternative definition of the Chebyshev polynomials, and the one we use here, is the following.

Definition 1. Let $\{T_n(x)\}_{n=0}^{\infty}$, the Chebyshev polynomials of the first kind, be the sequence of polynomials defined by the recurrence relation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad (1.3)$$

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for $n \geq 2$ and $T_0(x) = 1, T_1(x) = x$.

Definition 2. The sequence of Chebyshev polynomials of the second kind, $\{U_n(x)\}_{n=0}^{\infty}$, satisfies the recurrence

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad (1.4)$$

for $n \geq 2$, with $U_0(x) = 1$ and $U_1(x) = 2x$.

For a proof that these definitions are equivalent to the statements (1.1) and (1.2), see Rivlin (1990). Here is a list of the first few Chebyshev polynomials of each type:

n	$T_n(x)$	$U_n(x)$
0	1	1
1	x	$2x$
2	$2x^2 - 1$	$4x^2 - 1$
3	$4x^3 - 3x$	$8x^2 - 4x$
4	$8x^4 - 8x^2 + 1$	$16x^4 - 12x^2 + 1$
5	$16x^5 - 20x^3 + 5x$	$32x^5 - 32x^3 + 6x$

From these examples, certain patterns immediately appear. First, if $n \geq 0$ is even, $T_n(x)$ and $U_n(x)$ are even functions. Likewise, if n is odd, $T_n(x)$ and $U_n(x)$ are odd functions. Hence, $T_n(x) = (-1)^n T_n(-x)$ and $U_n(x) = (-1)^n U_n(-x)$ when $n > 0$. We shall soon explain this pattern and many more through our combinatorial interpretation

1.2 Weighted N -tilings

Our combinatorial interpretation of Chebyshev polynomials is motivated by a tiling interpretation of the Fibonacci numbers. Let $\{f_n\}$ be the Fibonacci sequence beginning with $f_0 = 1$ and $f_1 = 1$, with recurrence $f_n = f_{n-1} + f_{n-2}$. As we'll see, f_n is the number of ways to tile a one-dimensional length n board with squares (of length 1) and dominoes (of length 2). We call each of these ways of tiling a length n board an n -tiling.

Definition 3. An n -tiling is a sequence of squares (of length 1) and dominoes (of length 2) having total length n .

Why do the Fibonacci numbers correspond to the number of n -tilings? First look at the initial conditions. There is exactly one way of tiling a length

0 (i.e. empty) board: use no tiles. So $f_0 = 1$ makes sense. There is exactly one way of tiling a board of length 1: use one square. So, $f_1 = 1$. Now think of an arbitrary n -tiling where $n \geq 2$, of which there are f_n . Each n -tiling ends in a square or domino. If it ends in a square, remove the square and you are left with an $(n - 1)$ -tiling of which there are f_{n-1} . If it ends in a domino, remove the domino and you are left with an $(n - 2)$ -tiling of which there are f_{n-2} . Thus we get the recurrence $f_n = f_{n-1} + f_{n-2}$.

Recall from equation 1.4 that the Chebyshev polynomials of the second kind, $U_n(x)$, satisfy the recurrence

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x),$$

with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$. Clearly this recurrence does not beget as simple a tiling interpretation as the Fibonacci recurrence. However, if we allow our dominoes and squares to have *weights* then we can find a tiling interpretation. We call such a tiling a *weighted n -tiling* or when the context is clear just an *n -tiling*.

Definition 4. A weighted n -tiling is a sequence of light squares (of length 1), dark squares (of length 1) and dominoes (of length 2) having total length n . Light squares and dark squares each have weight x and dominoes have weight -1 .

We define the weight of a tiling to be the product of the weights of the tiles. For example, if we have the 5-tiling $\sigma =$ light square, dark square, domino, light square, then the weight of σ is

$$W(\sigma) = (x)(x)(-1)(x) = -x^3.$$

Observe that if an n -tiling contains s squares and d dominoes then $s + 2d = n$ and its weight is $x^s(-1)^d$. To make notation easier, we can write a for a light square, b for a dark square and D for a light domino, see Figure 1.1 below for an example. Thus the 5-tiling above in its word form is $\sigma = abDa$. In later proofs it will be necessary to introduce dark dominoes, which have weight -1 and are denoted \hat{D} .



Figure 1.1: This is a picture of the 5-tiling $\sigma =$ light square, dark square, domino, light square, denoted $abDa$.

Furthermore, we will refer to a square that can be either light or dark as just “ s ”, with weight $W(s) = 2x$. So $\sigma = bDs$ would represent two possible weighted n -tilings: $\sigma_1 = bDa$ and $\sigma_2 = bDb$. The following theorem states that $U_n(x)$ counts the weights of all of these weighted tilings.

Theorem 1. $U_n(x)$ is equal to the sum of the weights of all weighted n -tilings.

For example, $U_3(x)$ is the sum of the weights of all weighted 3-tilings. The weighted 3-tilings are $aaa, aab, aba, abb, baa, bab, bba, bbb, aD, bD, Da, Db$ as in Figure 1.2. The sum of the weights of these tilings is $8x^3 - 4x = U_3(x)$.

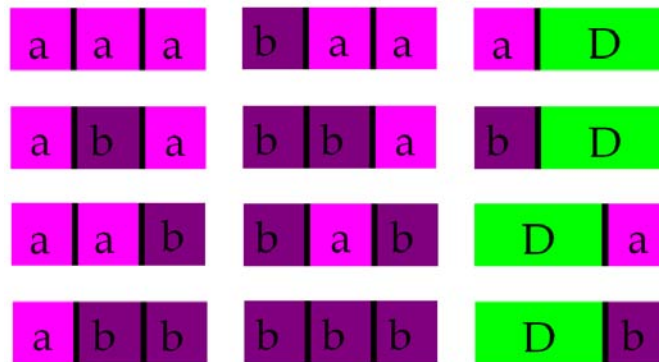


Figure 1.2: Here are all the weighted 3-tilings.

Proof. (By induction on the length of the word.)

First, the only 0-tiling (i.e. a tiling of the empty board) is the tiling that uses no squares and no dominoes. Thus we can think of its weight as $x^0(-1)^0 = 1$. Hence $U_0(x) = 1$ is correct. Second there are two 1-tilings: either a light square or dark square. Thus the total weight of all 1-tilings is $x + x = 2x$. Hence $U_1(x) = 2x$ is correct.

Now, assume that for $k = n - 1$ and $k = n - 2$ the sum of all weighted k -tilings is $U_k(x)$. Consider all unrestricted n -tilings. Either they end with a light square, a dark square, or a domino. What is the total weight of those that end with a light square? The sum of the weights of all tilings ending with a light square is just x (the weight of the light square) times the total weight of the remaining section of length $n - 1$. By our induction hypothesis, that weight is $U_{n-1}(x)$. Thus the answer is $xU_{n-1}(x)$. The total

weight of those that end with a dark square is also $xU_{n-1}(x)$. The total weight of those that end in a domino is just (-1) , the weight of the domino, times the total weight of the remaining $(n - 2)$ -tilings, which is $U_{n-2}(x)$. Thus the total weight of all n -tilings is

$$\begin{aligned} xU_{n-1}(x) + xU_{n-1}(x) + (-1)U_{n-2}(x) &= 2xU_{n-1}(x) - U_{n-2}(x) \\ &= U_n(x), \end{aligned}$$

where the last equality is the recurrence satisfied by the $U_n(x)$. By the principle of induction, our claim is true for all non-negative integers n . \square

Furthermore, we have a similar interpretation of Chebyshev polynomials of the first kind. Notice that both kinds of Chebyshev polynomials satisfy the same recurrence; the only difference is the initial conditions. Because $T_1(x) = x$, we require that if the first tile in a tiling is a square, then it must be light. Thus our interpretation of Chebyshev polynomials of the first kind is the same as for the second kind, but we add the restriction that the initial tile cannot be a dark square. Arguing as in the proof of Theorem 1, we have

Theorem 2. $T_n(x)$ is equal to the sum of the weights of those weighted n -tilings beginning with a domino or light square. We often call these restricted tilings.

For example, $T_3(x)$ is the sum of the weights of all restricted 3-tilings. The restricted 3-tilings are $aaa, aab, aba, abb, aD, Da, Db$. The sum of the weights of these tilings is $4x^3 - 3x = T_3(x)$. Occasionally, we will also exploit the following immediate corollary, which has the alternate restriction that requires that the initial tile cannot be a light square:

Corollary 1. $T_n(x)$ is equal to the sum of the weights of those weighted n -tilings beginning with a domino or **dark** square.

It will be useful in many proofs to refer to the set of tilings of a length n board.

Definition 5. Let \mathcal{U}_n be the set of all weighted n -tilings and \mathcal{T}_n be the set of all weighted n -tilings where an initial square must be light (i.e. all restricted n -tilings).

1.3 Combinatorial Proofs of Basic Properties

In this section we present combinatorial proofs of some of the basic facts about Chebyshev polynomials. One of the big advantages of the combinatorial interpretation is that proofs give insight into why some of these basic facts hold, they are not just a mess of algebra.

The following formulas give ways of computing the coefficients of $T_n(x)$ and $U_n(x)$. These formulas can easily be constructed from looking at the first few polynomials of each kind, but proving them algebraically is a little more difficult. On the other hand, proving them using our tiling interpretation will be trivial.

Proposition 1. $T_n(x) = t_0^{(n)} + t_1^{(n)}x + \dots + t_n^{(n)}x^n$ and $U_n(x) = u_0^{(n)} + u_1^{(n)}x + \dots + u_n^{(n)}x^n$, where

$$u_{n-2k}^{(n)} = (-1)^k 2^{n-2k} \binom{n-k}{k} \quad (1.5)$$

$$t_{n-2k}^{(n)} = (-1)^k 2^{n-2k-1} \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right], \quad (1.6)$$

for $k = 0, \dots, \lfloor \frac{n-1}{2} \rfloor$. All other coefficients are zero.

Proof. First we prove 1.5. $u_{n-2k}^{(n)}x^{n-2k}$ is the sum of weights of all weighted n -tilings having k dominoes. Such tilings must have $n - 2k$ squares and $n - k$ total tiles. There are $\binom{n-k}{k}$ ways to pick which k of the $n - k$ tiles are dominoes. The remaining $n - 2k$ tiles are all squares. Since we are working with unrestricted tilings, there are 2^{n-2k} ways to decide whether each square is light or dark. The weight of each tiling is $(-1)^k x^{n-2k}$. Hence

$$u_{n-2k}^{(n)}x^{n-2k} = (-1)^k x^{n-2k} 2^{n-2k} \binom{n-k}{k},$$

which, upon cancelling the x^{n-2k} , yields our result.

Now we prove 1.6. $t_{n-2k}^{(n)}x^{n-2k}$ is the sum of the weights of those restricted n -tilings having k dominoes. Such tilings have $n - 2k$ squares and $n - k$ total tiles. There are 2^{n-2k-1} ways to color squares 2 through $n - 2k$ light or dark. Now we condition on the color of the first square (not to be confused with the first tile). Suppose the first square is dark. Since a restricted tiling cannot begin with a dark square, we know our tiling must start with a domino. There are $\binom{n-k-1}{k-1}$ ways to choose which of the $n - k - 1$

remaining tiles are the $k - 1$ dominoes. On the other hand, suppose the first square is light. Then our tiling can start with a domino or a square. Thus there are $\binom{n-k}{k}$ ways to decide which tiles are dominoes. The weight of each tiling is $(-1)^k x^{n-2k}$. Hence

$$t_{n-2k}^{(n)} x^{n-2k} = (-1)^k x^{n-2k} 2^{n-2k-1} \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right],$$

which, when we cancel the x^{n-2k} , gives our result for $n - 2k$ coefficients.

Notice that we have only covered half of the coefficients; we still need to compute the $n - (2k + 1)$ coefficients. Observe that if the tiling has length n , the number of squares in the tiling must be of the form $n - 2k$, where k is the number of dominoes. Therefore there are no n -tilings containing exactly $n - (2k + 1)$ squares. Hence

$$t_{n-(2k+1)}^{(n)} = u_{n-(2k+1)}^{(n)} = 0,$$

for $k = 0, \dots, \lfloor \frac{n-1}{2} \rfloor$. □

Alternatively, instead of indexing by the number of dominoes, k , we could have indexed by the number of squares, j . Using the change of variables $j = n - 2k$ or equivalently $k = (n - j)/2$, we get the following corollary. This form of the previous result allows us to give the value of the coefficient $u_j^{(n)}$ directly, instead using the more cumbersome $u_{n-2k}^{(n)}$.

Corollary 2. Let $T_n(x) = t_0^{(n)} + t_1^{(n)}x + \dots + t_n^{(n)}x^n$ and $U_n(x) = u_0^{(n)} + u_1^{(n)}x + \dots + u_n^{(n)}x^n$. If j and n have the same parity, then

$$u_j^{(n)} = (-1)^{\frac{n-j}{2}} 2^j \binom{(n+j)/2}{j} \quad (1.7)$$

$$t_j^{(n)} = (-1)^{\frac{n-j}{2}} 2^{j-1} \left[\binom{(n+j)/2}{j} + \binom{(n+j)/2-1}{j} \right], \quad (1.8)$$

If j has the opposite parity of n then $t_j^{(n)} = 0$ and $u_j^{(n)} = 0$.

Here is another formula for the coefficients of the n th Chebyshev polynomial of the first kind:

Proposition 2.

$$T_n(x) = t_0^{(n)} + t_1^{(n)}x + t_2^{(n)}x^2 + \dots + t_n^{(n)}x^n,$$

with

$$\begin{aligned} t_{n-(2k+1)}^{(n)} &= 0, & k &= 0, \dots, \lfloor \frac{n-1}{2} \rfloor \\ t_{n-2k}^{(n)} &= (-1)^k \sum_{j=k}^{\lfloor n/2 \rfloor} \binom{n}{2j} \binom{j}{k}, & k &= 0, \dots, \lfloor \frac{n}{2} \rfloor. \end{aligned} \quad (1.9)$$

At first this formula may seem daunting, but using the tiling interpretation it is possible to understand why such a formula holds: Consider all restricted weighted n -tilings. Let k denote the number of dominoes in such a tiling. Then there are $n - 2k$ squares.

Now we define a rather interesting term.

Definition 6. For a restricted tiling, we let an interval of darkness be a section of the tiling of the form Db^k or ab^{k+1} for some integer $k \geq 0$.

For example, the 8-tiling $abDbbba$ has exactly two intervals of darkness, ab and $Dbbb$. We say denote the endpoints of the intervals ab and $Dbbb$, by $(1, 2)$ and $(3, 7)$ respectively. The 12-tiling $DabaaabbDb$ has 4 intervals of darkness, D , ab , abb , and Db , with respective endpoints $(1, 2)$, $(3, 4)$, $(7, 9)$, $(10, 12)$.

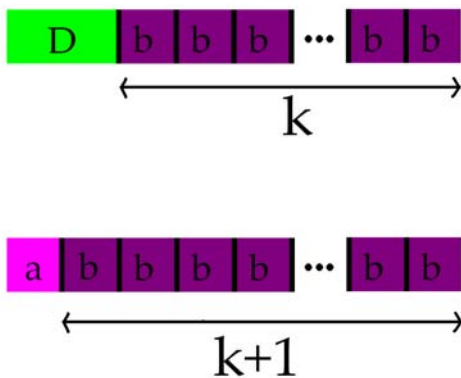


Figure 1.3: Here are the two types of intervals of darkness: Db^k and ab^{k+1} , for $k \geq 0$.

Notice that two intervals of darkness never overlap. Furthermore, any tile not in an interval of darkness must be a light square, since all dominoes and dark squares are contained in an interval of darkness. So, specifying the location of the intervals of darkness and whether each begins D or

ab uniquely defines a restricted tiling. Likewise, it is easy to see that every restricted tiling can be decomposed into intervals of darkness and light squares filling in the gaps. Therefore we can uniquely and completely characterize restricted tilings based on the location of intervals of darkness and whether each interval begins D or ab .

Now we are ready for the combinatorial proof.

Proof. Question: What is the weight of all restricted n -tilings using k dominoes?

A1: $t_{n-2k}^{(n)}x^{n-2k}$, the $n - 2k$ term in the polynomial $T_n(x)$.

A2: Enumerate the tilings by letting j be the number of intervals of darkness. (Note: since there are k dominoes, there must be at least k intervals. Also there can be at most $\lfloor n/2 \rfloor$ intervals, since each interval has length ≥ 2 .) First choose the endpoints of the j intervals of darkness. There are $\binom{n}{2j}$ ways to choose the endpoints. Next, we know that k of these intervals of darkness start with a domino and the remaining $j - k$ start with light square dark square. So there are $\binom{j}{k}$ ways to choose which intervals start with a domino. The weight of each tiling is $(-1)^k x^{n-2k}$. Hence, the second answer is

$$(-1)^k x^{n-2k} \sum_{j=k}^{\lfloor n/2 \rfloor} \binom{n}{2j} \binom{j}{k}.$$

Thus equating the answers and cancelling the powers of x we see that

$$t_{n-2k}^{(n)} = (-1)^k \sum_{j=k}^{\lfloor n/2 \rfloor} \binom{n}{2j} \binom{j}{k},$$

for $k = 0, \dots, \lfloor \frac{n}{2} \rfloor$.

We have addressed only half of the coefficients, those of the form $t_{n-2k}^{(n)}$. Notice that if the tiling is of length n , then the number of squares in the tiling must be of the form $n - 2k$, where k is the number of dominoes. Therefore there are no n -tilings containing exactly $n - (2k + 1)$. Hence

$$t_{n-(2k+1)}^{(n)} = 0,$$

for $k = 0, \dots, \lfloor \frac{n-1}{2} \rfloor$. □

Chapter 2

Chebyshev Identities

2.1 Elementary Identities

In this section we present combinatorial proofs of some of the basic Chebyshev identities. The following proofs involve asking a combinatorial question and answering it in two different ways.

2.1.1 Converting from $T_n(x)$ to $U_n(x)$

The first two identities demonstrate some of the most fundamental relationships between Chebyshev polynomials of the first and second kinds.

Theorem 3. $U_n(x) = T_n(x) + xU_{n-1}(x)$, for $n \geq 1$.

Proof. What is the sum of the weights of all weighted n -tilings?

A1: By Theorem 1, this is just $U_n(x)$.

A2: Alternatively, the total weight of those n -tilings where an initial tile can be a domino or light square is $T_n(x)$ by Theorem 2. The only tilings we have left out are those beginning with a dark square. The total weight of such tilings is $xU_{n-1}(x)$, since the weight of a dark colored square is x and the total weight of tiling the remaining $n - 1$ spots where there are no restrictions is $U_{n-1}(x)$. Hence the total weight of such tilings is $T_n(x) + xU_{n-1}(x)$.

Equating these answers gives us $U_n(x) = T_n(x) + xU_{n-1}(x)$. □

This next theorem demonstrates another fundamental relationship between the two kinds of Chebyshev polynomials.

Theorem 4. For $n \geq 2$, $2T_n(x) = U_n(x) - U_{n-2}(x)$.

Proof. What is the sum of the weights of all weighted n -tilings where the first tile (whether it is domino or square) can be colored light or dark?

A1: First observe that the total weight of the tilings with the first tile light is $T_n(x)$. Currently our interpretation of $T_n(x)$ requires an initial square to be light, but does not assign a color to dominoes. So, without loss of generality, in each of the n -tilings we can color an initial domino light. By similar logic, the total weight of the tilings starting with a dark first tile is $T_n(x)$ also. Thus one answer is $2T_n(x)$.

A2: Alternatively, what's the total weight of those n -tilings where an initial tile can be a light domino, light square, or dark square? By Theorem 1 that is $U_n(x)$. The only tilings we have left out are those beginning with a dark domino. The total weight of such tilings is $(-1)U_{n-2}(x)$, since the weight of a dark colored domino is (-1) and the total weight of tiling the remaining $n - 2$ spots where there are no restrictions is $U_{n-2}(x)$. Hence the total weight is $U_n(x) - U_{n-2}(x)$.

Equating these answers gives us $2T_n(x) = U_n(x) - U_{n-2}(x)$. \square

2.1.2 The String of Lights

The next identity we prove shows how the sum of a series of first kind polynomials, is merely a polynomial of the second kind. We call it "The String of Lights" because its proof involves enumerating tilings based on the length of the initial string of j light squares.

Theorem 5. $U_n(x) = \sum_{j=0}^n x^j T_{n-j}(x)$.

There are multiple ways to prove this identity. Proof by induction can be done easily from Theorem 3, however here we can present an equally simple, but more enlightening, combinatorial proof.

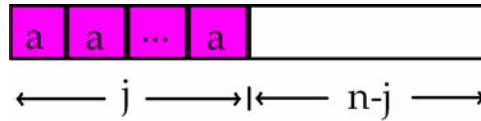


Figure 2.1: Every restricted tiling must start with j light squares, where $0 \leq j \leq n$.

Proof. What is the sum of the weights of all weighted n -tilings?

A1: By Theorem 1, this is just $U_n(x)$.

A2: What is the total weight of such tilings that start with j light squares before the first domino or dark square? The weight of the first j light squares is x^j and the total weight of the remaining $(n - j)$ -tilings that must start with a domino or dark square is just $T_{n-j}(x)$, by Corollary 1. Summing over all possible numbers of initial lights squares yields $\sum_{j=0}^n x^j T_{n-j}(x)$. \square

The identity above demonstrates how $U_n(x)$ can be written as the sum of the $x^j T_{n-j}$. The next identity shows how $T_n(x)$ can be written in a similar manner.

Theorem 6. For $n \geq 2$, $T_n(x) = -U_{n-2}(x) + \sum_{j=1}^n x^j T_{n-j}(x)$.

Proof. What is the sum of the weights of all restricted n -tilings?

A1: By Theorem 2, this is just $T_n(x)$.

A2: What is the total weight of those n -tilings that start with j light squares before the first domino or dark square? If $j = 0$, then the tilings must start with a domino (of weight -1), since restricted tilings start with either a light square or domino. In that case, the weight of the remaining length $n - 2$ tiling is $U_{n-2}(x)$. If $1 \leq j \leq n$, the weight of the first j light squares is x^j and the total weight of the remaining $(n - j)$ -tilings that must start with a domino or dark square is, by Corollary 1, $T_{n-j}(x)$. Summing over all possible j yields

$$(-1)U_{n-2}(x) + \sum_{j=1}^n x^j T_{n-j}(x).$$

\square

2.1.3 Determinant Identities

The determinant identities satisfied by the Chebyshev polynomials are beautiful, but different from the identities we have seen so far. The first and second kind of Chebyshev polynomials each satisfy a slightly different determinant identity. As we will see, the identity satisfied by the second kind is more elegant than that satisfied by the first, a direct reflection of the relative simplicity of their combinatorial interpretations.

Theorem 7. For $n \geq 1$,

$$T_n(x) = \det \begin{bmatrix} x & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2x & 1 & 0 & \ddots & 0 & 0 \\ 0 & 1 & 2x & 1 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 2x & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2x \end{bmatrix},$$

where the matrix is $n \times n$.

Theorem 8. For $n \geq 1$,

$$U_n(x) = \det \begin{bmatrix} 2x & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2x & 1 & 0 & \ddots & 0 & 0 \\ 0 & 1 & 2x & 1 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 2x & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2x \end{bmatrix},$$

where the matrix is $n \times n$.

One way to prove these identities is inductively using cofactor expansion on the bottom row. However, we prove these identities using a combinatorial interpretation of determinants involving weighted directed bipartite graphs, as given in Aigner (2001). Let $M = [m_{ij}]$ be an $n \times n$ matrix.

Create vertex sets $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$ and $\mathcal{B} = \{B_1, B_2, \dots, B_n\}$. For all i and j draw a directed edge from A_i to B_j and give it weight m_{ij} .

Consider a bijection (or perfect matching) f between \mathcal{A} and \mathcal{B} . This bijection can be thought of as a permutation σ of the indices, such that $f(A_i) = B_{\sigma(i)}$. We define a *vertex-disjoint path system* \mathcal{P}_σ from \mathcal{A} to \mathcal{B} as the set of directed edges $\mathcal{P}_\sigma = \{(A_1, B_{\sigma(1)}), \dots, (A_n, B_{\sigma(n)})\}$. The weight of such a system is the product of the weights of each path

$$w(\mathcal{P}_\sigma) = w((A_1, B_{\sigma(1)})) \cdots w((A_n, B_{\sigma(n)})) = \prod_{i=1}^n m_{i, \sigma(i)}.$$

Hence

$$\det(M) = \sum_{\sigma \in \Pi_n} \text{sign}(\sigma) m_{1, \sigma(1)} m_{2, \sigma(2)} \cdots m_{n, \sigma(n)}, \quad (2.1)$$

$$= \sum_{\sigma \in \Pi_n} \text{sign}(\sigma) \prod_{i=1}^n m_{i, \sigma(i)} \quad (2.2)$$

$$= \sum_{\sigma \in \Pi_n} \text{sign}(\sigma) w(\mathcal{P}_\sigma), \quad (2.3)$$

where $\text{sign}(\sigma)$ is the sign of the permutation σ and Π_n is the set of all length n permutations. Thus the combinatorial interpretation is that the determinant counts the signed sum of the weights of vertex-disjoint path systems.

Proof. (of Theorem 8)

This is the easier case. Let $M = [m_{ij}]$ be the matrix given in Theorem 8. Figure 2.2 on the next page is a picture of the directed bipartite graph given in the interpretation. Notice that we do not include edges with weight 0 since any path systems using these edges have weight 0 and therefore contribute nothing to the determinant.

We prove the result by induction. Let $n = 1$. Then there is only one possible path system $\mathcal{P}_\sigma = \{(A_1, B_1)\}$. Thus

$$w(\mathcal{P}_\sigma) = m_{1,1} = 2x = U_1(x).$$

Let $n = 2$. Then there are two possible path systems, $\{(A_1, B_1), (A_2, B_2)\}$, $\{(A_1, B_2), (A_2, B_1)\}$, corresponding to the two possible permutations, id and (12) . The signed weights of these are $(2x)(2x) = 4x^2$ and $-(1)(1) = -1$ respectively. (Note that the second weight is negative, since (12) is an odd permutation.) Thus sum of the signed weights is

$$4x^2 - 1 = U_2(x).$$

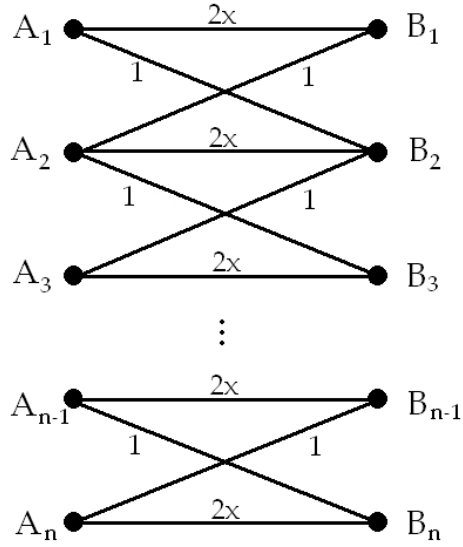


Figure 2.2: This is the weighted bipartite graph on \mathcal{A} and \mathcal{B} .

Now assume that our claim holds for n and $n - 1$. Consider the $n + 1$ case. A size $n + 1$ path system with non-zero weight must have the edge (A_n, B_n) or the edges (A_{n-1}, B_n) and (A_n, B_{n-1}) . In the first case, the total signed weight is $2x$ times the signed weight of the remaining size n path system. In the second case, the total signed weight is $-(1)(1) = -1$ times the weight of the remaining size $n - 1$ path system. (Again, note the negative sign since the second case changes the parity of the of the permutation.) Therefore we have the sum of the signed weights is

$$2xU_n(x) - U_{n-1}(x) = U_{n+1}(x).$$

□

Proof. (of Theorem 7)

Applying the same reasoning as in Case 1, the only difference is in the initial conditions. When $n = 1$, there is only one possible path system $\mathcal{P}_\sigma = \{(A_1, B_1)\}$. Thus

$$w(\mathcal{P}_\sigma) = m_{1,1} = x = T_1(x).$$

When $n = 2$, there are two possible path systems, $\{(A_1, B_1), (A_2, B_2)\}$, $\{(A_1, B_2), (A_2, B_1)\}$, corresponding to the two possible permutations \mathbf{id} , (12).

The signed weights of these are $(x)(2x) = 2x^2$ and $-(1)(1) = -1$ respectively. (Note that the second weight is negative, since (12) is an odd permutation.) Thus the sum of the signed weights is

$$2x^2 - 1 = T_2(x).$$

The induction step is the same as the previous proof. \square

2.2 Advanced Identities

The next few identities require more difficult proof techniques. Instead of just asking combinatorial questions, these proofs rely on bijections and other methods

2.2.1 The Five Part Identity

The following theorem has a more interesting and involved proof. Notice the similarities between this identity and Theorem 4. The Five Part Identity gets its name because each side of the equality is a different way of counting five different groups of tilings.

Theorem 9. $T_n(x) - T_{n-2}(x) = 2(x^2 - 1)U_{n-2}(x)$, for $n \geq 2$.

Proof. What is the sum of the weights of all weighted n -tilings so that the first tile can be either light or dark, with the restriction that if the initial tile is dark it must be followed by a domino?

A1: Note that the possible beginnings of the word form of such tilings are as , D , aD , bD , and $\hat{D}D$. (Like before, a , b , s , D , \hat{D} respectively denote a light square, a dark square, a square of either color, a light domino, and a dark domino. Note that a dark domino can only appear in the first position.) What is the total weight of those tilings that begin with a light square or a light domino? By Theorem 2 the answer is just $T_n(x)$. What is the total weight of those tilings that begin with a dark tile? Such a tiling can be created by taking a restricted $(n - 2)$ -tiling, changing the color of the first tile to dark, then inserting a light domino (of weight -1) as our second tile. Hence these dark tilings have weight $-T_{n-2}(x)$. Altogether, our total weight is

$$T_n(x) - T_{n-2}(x).$$

A2: What is the total weight of such tilings with word form that starts as ? Well $W(as) = (x)(x+x) = 2x^2$ and the remaining $n-2$ spaces have total weight $U_{n-2}(x)$. Hence these have total weight $2x^2U_{n-2}(x)$. What is the total weight of those tilings starting with a light colored domino (denoted D)? $W(D) = -1$ and the remaining spaces have weight $U_{n-2}(x)$. Hence these have total weight $-U_{n-2}(x)$. What is the total weight of those tilings that start with aD , bD , or $\hat{D}D$? Remove the D occurring after the first tile from each of these words and we are left with length $n-2$ words that start with a , b , or \hat{D} . The total weight of all $(n-2)$ -tilings that start with a , b , or \hat{D} is just the total weight of all weighted $(n-2)$ -tilings, which is $U_{n-2}(x)$. Since the removed domino had weight -1 , the total weight of these tilings is $-U_{n-2}(x)$. Summing up over all five cases, we have the total weight

$$\begin{aligned} 2x^2U_{n-2}(x) - U_{n-2}(x) - U_{n-2}(x) \\ = 2(x^2 - 1)U_{n-2}(x). \end{aligned}$$

Equating answers A1 and A2 yields

$$T_n(x) - T_{n-2}(x) = 2(x^2 - 1)U_{n-2}(x).$$

□

2.2.2 Swapping Tails

To prove the following identities we introduce the technique of *tailswapping*. First, we must introduce the concept of a tiling being *breakable*.

Definition 7. We say that a tiling is m -breakable or breakable at cell m if there is no domino covering both cells m and $m+1$.

The intuitive idea is that an m -breakable tiling can be separated into two distinct tilings with the separation occurring between cells m and $m+1$. Note that the weight of all m -breakable $m+n$ -tilings is $U_m(x)U_n(x)$. If the tilings are restricted, then the total weight is $T_m(x)U_n(x)$. The number of m -unbreakable tilings is $-U_{m-1}(x)U_{n-1}(x)$, since there is a domino of weight -1 covering cells m and $m+1$. Likewise the number of restricted m -unbreakable tilings is $-T_{m-1}(x)U_{n-1}(x)$. Observing that each length $m+n$ tiling is either breakable or unbreakable at cell m immediately gives us the following two identities.

Theorem 10. For integers $m, n \geq 1$,

$$U_{m+n}(x) = U_m(x)U_n - U_{m-1}(x)U_{n-1}(x)$$

and

$$T_{m+n}(x) = T_m(x)U_n - T_{m-1}(x)U_{n-1}(x).$$

This next theorem uses the idea of breakability as well as the more advanced technique of tailswapping.

Theorem 11. If m, n are nonnegative integers and $n \geq m$ then

$$2T_n(x)T_m(x) = T_{n+m}(x) + T_{n-m}(x). \tag{2.4}$$

Proof. Here we do a bijective proof showing a 1-to-2 correspondence between $\mathcal{T}_n \times \mathcal{T}_m$ and $\mathcal{T}_{n+m} \cup \mathcal{T}_{n-m}$. That is, each element $x \in \mathcal{T}_n \times \mathcal{T}_m$ is mapped to two elements $\phi_1(x), \phi_2(x) \in \mathcal{T}_{n+m} \cup \mathcal{T}_{n-m}$, such that every element of $\mathcal{T}_{n+m} \cup \mathcal{T}_{n-m}$ is hit exactly once. First observe that if $m = 0$ the result is trivial, so we assume $m \geq 1$.

For (A, B) in $\mathcal{T}_n \times \mathcal{T}_m$, put

$$\phi_1(A, B) = AB,$$

where AB is the tiling created by concatenating A with B . Observe that $AB \in \mathcal{T}_{n+m}$. Furthermore, because AB is the concatenation of elements in \mathcal{T}_n and \mathcal{T}_m , AB is breakable at cell n , and cell $n + 1$ has a light square or the start of a domino.

$\phi_2(A, B)$ depends on the initial tile of B . If B begins with a light square (we say $B = aB'$), then

$$\phi_2(A, B) = AbB'.$$

Observe that $AbB' \in \mathcal{T}_{n+m}$, AbB' is breakable at n , and has a dark square in cell $n + 1$. Since all elements in \mathcal{T}_{n+m} that are n -breakable have a light square, the start of a domino, or a dark square at cell $n + 1$, we have hit all n -breakable elements of \mathcal{T}_{n+m} . It remains to hit the elements of \mathcal{T}_{n+m} which are not n -breakable as well as all of the elements of \mathcal{T}_{n-m} .

Now, the final case is when B begins with a domino (i.e $B = DB''$). In this case let B^{-1} be the reversal of B . So then $B^{-1} = (B'')^{-1}D$. Now we tailswap. (See Figure 2.3 for a visual explanation.) Line up the tilings A and B^{-1} so that they are almost right justified, but so that B^{-1} extends one space past A . Now let $A = A_1A_2$ and $B^{-1} = B_1B_2D$ defined by the

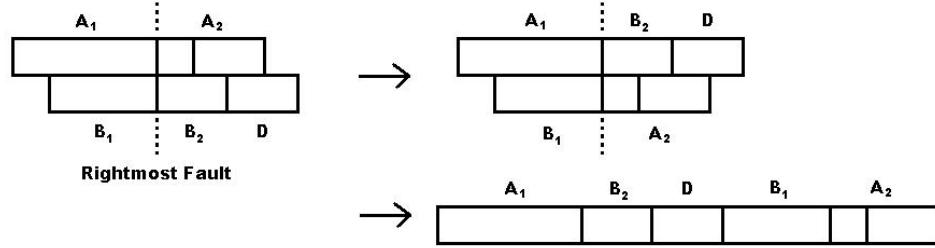


Figure 2.3: To Tailswap: First, line up tilings A and B^{-1} so that the right end of B^{-1} extends one space past the right end of A . Second, locate the rightmost fault line, the place where this arrangement of tilings can be broken. Swap the tails of the two tilings. Finally, append the bottom tiling to the end of the top tiling to get the result.

rightmost fault line (the rightmost place where the arrangement of tilings can be broken). If A and B are tailswappable, then we let

$$\phi_2(A, B) = \phi_2(A_1 A_2, (B_1 B_2 D)^{-1}) = A_1 B_2 D B_1 A_2.$$

Note that if tailswapping is possible, then $\phi_2(A, B) = A_1 B_2 D B_1 A_2$ is an element of \mathcal{T}_{n+m} that is not n -breakable, since $A_1 B_2 D$ has length $n + 1$, with a domino covering cells n and $n + 1$.

If m is even, the only pairs (A, B) that cannot be tailswapped are those where B is all dominoes ($m/2$ of them) and A ends in $m/2$ dominoes. Notice that all of these have the form $(A' D^{m/2}, D^{m/2})$. In this case we let

$$\phi_2(A' D^{m/2}, D^{m/2}) = A'.$$

Where $A' \in \mathcal{T}_{n-m}$. Thus, if m is even, we have hit everything in $\mathcal{T}_{n+m} \cup \mathcal{T}_{n-m}$.

On the other hand if m is odd, all pairs (A, B) can be tailswapped. However, not all elements of \mathcal{T}_{n+m} will be hit by the tailswapping, meaning that our two maps ϕ_1 and ϕ_2 give are not surjective. The elements that will not be hit are those elements of \mathcal{T}_{n+m} that end in m dominoes and all elements of \mathcal{T}_{n-m} . The weight of all the elements in \mathcal{T}_{n+m} ending in m dominoes is just $(-1)^m T_{n+m-2m}(x) = -T_{n-m}(x)$, since m is odd. The weight of these elements is exactly cancelled out by the weight of the elements in \mathcal{T}_{n-m} . So the total weight of all elements not hit by our two maps is 0. Thus our map,

while not a bijection when m is odd, still gives us the result

$$2T_n(x)T_m(x) = T_{n+m}(x) + T_{n-m}(x). \tag{2.5}$$

□

2.2.3 The Cancellation Identity

Here we introduce the idea of the domino conjugate of a tiling.

Definition 8. Let $2\mathcal{T}_n$ denote the set of weighted n -tilings where the first tile can be either a light square, dark square, light domino, or dark domino, but other dark dominoes are not allowed. For $A \in 2\mathcal{T}_n$, if A does not begin with a dark domino, its domino conjugate is $\hat{D}A$; if A begins with a dark domino ($A = \hat{D}A''$), then its domino conjugate is A'' .

In other words, the domino conjugate of a tiling is simply the tiling created by adding a dark domino at the beginning if there wasn't one before, or removing it if there is one. Since dark dominoes can only occur as an initial tile, if we let f map a tiling to its domino conjugate, then $f(f(A)) = A$. Hence f is an involution.

Theorem 12. For $n \geq 0$

$$(1 + x)U_n(x) = 1 + T_{n+1}(x) + \sum_{i=1}^n 2T_i(x). \tag{2.6}$$

Proof. In this proof, we show that every tiling on the right hand side of the equality cancels with its domino conjugate except for those tilings on the left side (hence "The Cancellation Identity").

Observe that the right side of the equation is the sum of the weights of the tilings in the set $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_{n+1} \cup \bigcup_{i=1}^n 2\mathcal{T}_i$. Next, suppose we let f be the map that sends a tiling to its domino conjugate. Since f either causes a domino to be removed or to be added, $w(f(A)) = (-1)w(A)$. Hence $w(A) + w(f(A)) = 0$, meaning f is a *sign-reversing* involution. Thus if A and $f(A)$ are both in \mathcal{T} the net contribution of their weights is 0.

Normally, we let \mathcal{T}_{n+1} denote the set of $(n + 1)$ -tilings where an initial square must be light and all dominoes must be light. For this proof, we use the alternative definition. Let \mathcal{T}_{n+1} be the set of weighted $(n + 1)$ -tilings where an initial square must be light and an initial domino is dark (all other dominoes are still light). This change is minor, but useful to the proof.

Many of the tilings in \mathcal{T} also have their domino conjugate in \mathcal{T} . The combined weight of all such elements is zero. So, it remains to identify

which tilings do not have their domino conjugate and calculate the combined weight of those tilings.

Suppose $A \in 2\mathcal{T}_i$ for some $0 \leq i \leq n-1$. If A begins with a dark domino, $A = \hat{D}A''$, then A has length at least 2, so $f(A) = A''$ is well defined. Otherwise, $f(A) = \hat{D}A$ has length at most $n+1$ and is in \mathcal{T} . Thus for all such $A \in \mathcal{T}$, we have $f(A) \in \mathcal{T}$.

Now suppose $A \in 2\mathcal{T}_n$. Then, $f(A)$ is only in \mathcal{T} if A begins with a dark domino, since otherwise $f(A)$ has length $n+2$. Thus f cannot be applied to any tilings that begin with a light domino or a light or dark square. Such tilings have weight $U_n(x)$.

Finally suppose $A \in \mathcal{T}_{n+1}$, then A begins with a dark domino or light square. If A begins with a dark domino, $f(A) \in \mathcal{T}$. Otherwise A begins with a light square, in which case $f(A)$ has length $n+3$ so $f(A) \notin \mathcal{T}$. Hence f is undefined for tilings of the form aa' , where $A' \in \mathcal{U}_n$. The total weight of these tilings is $xU_n(x)$.

Altogether, the total weight of the tilings in \mathcal{T} is equal to

$$U_n(x) + xU_n(x) = (1+x)U_n(x),$$

as claimed. □

2.2.4 The Reduction Identity

In the proof of this next identity, we will use tailswapping to get a near correspondence between $\mathcal{T}_{n+m} \times \mathcal{T}_{n-m}$ and $\mathcal{T}_n \times \mathcal{T}_n$. Then what remains reduces to a second identity. Then, by proving the second identity, we will have proved both identities.

Theorem 13. *Let nonnegative integers n, m , such that $n \geq m$. Then*

$$T_n^2(x) + T_m^2(x) = 1 + T_{n+m}(x)T_{n-m}(x).$$

Proof. First we show the near correspondence between $\mathcal{T}_{n+m} \times \mathcal{T}_{n-m}$ and $\mathcal{T}_n \times \mathcal{T}_n$. Let (A, B) be an element of $\mathcal{T}_{n+m} \times \mathcal{T}_{n-m}$. Align the two tilings so that the $(n-m)$ -tiling starts at cell $m+1$ of the $(n+m)$ -tiling. Then, if possible, tailswap. This will create n -tilings C and D . Let (C, D) be the corresponding element of $\mathcal{T}_n \times \mathcal{T}_n$. We can go backwards, starting with an element (C, D) in $\mathcal{T}_n \times \mathcal{T}_n$, aligning the tilings so that the second n -tiling starts at cell $m+1$, and tailswapping to get the two n -tilings A and B , which make the corresponding element $(A, B) \in \mathcal{T}_{n+m} \times \mathcal{T}_{n-m}$.

Now that we have a near 1-1 correspondence by tailswapping, we must determine which tilings cannot be tailswapped or cannot be hit by tailswapping.

Case 1: $n - m$ is odd.

Since $n - m$ is odd, every element $(A, B) \in \mathcal{T}_{n+m} \times \mathcal{T}_{n-m}$ can be tailswapped to form an element $(C, D) \in \mathcal{T}_n \times \mathcal{T}_n$. However, not all elements in $(C, D) \in \mathcal{T}_n \times \mathcal{T}_n$ have preimages in $\mathcal{T}_{n+m} \times \mathcal{T}_{n-m}$ under tailswapping. There are two cases where this occurs. In the first case, C begins with a restricted $(m - 1)$ -tiling, followed by $(n - m + 1)/2$ dominoes and D begins with $(n - m + 1)/2$ dominoes, followed by an unrestricted $(m - 1)$ -tiling. In the second case, C begins with a restricted m -tiling, followed by a dark square, followed by $(n - m - 1)/2$ dominoes and D begins with $(n - m + 1)/2$ dominoes, followed by an unrestricted $(m - 1)$ -tiling. The total weight of all these tilings is

$$\begin{aligned} T_{m-1}(x)(-1)^{n-m+1}U_{m-1}(x) + T_m(x)x(-1)^{n-m}U_{m-1}(x) \\ = (T_{m-1}(x) - xT_m(x))U_{m-1}(x) \end{aligned}$$

Case 2: $n - m$ even. Since $n - m$ is even, every element of $\mathcal{T}_n \times \mathcal{T}_n$ has a preimage in $\mathcal{T}_{n+m} \times \mathcal{T}_{n-m}$ under tailswapping. However, some elements $(A, B) \in \mathcal{T}_{n+m} \times \mathcal{T}_{n-m}$ are not tailswappable or have an image that is not in $\mathcal{T}_n \times \mathcal{T}_n$. These elements come in two cases. In the untailswappable case, A begins with a restricted $(m - 1)$ -tiling, followed by $(n - m + 2)/2$ dominoes, followed by an unrestricted $(m - 1)$ -tiling and B is just $(n - m)/2$ dominoes. In the second case, A begins with a restricted m -tiling, followed by a dark square, followed by $(n - m)/2$ dominoes, followed by an unrestricted $(m - 1)$ -tiling and B is again just $(n - m)/2$ dominoes. The weight of all these tilings is

$$\begin{aligned} T_{m-1}(x)(-1)^{n-m+1}U_{m-1}(x) + T_m(x)x(-1)^{n-m}U_{m-1}(x) \\ = (xT_m(x) - T_{m-1}(x))U_{m-1}(x). \end{aligned}$$

Thus we have shown (in both the $n - m$ even and odd cases) that the weight of the elements in $\mathcal{T}_{n+m} \times \mathcal{T}_{n-m}$ minus the weight of the elements in $\mathcal{T}_n \times \mathcal{T}_n$ is

$$T_{n+m}(x)T_{n-m}(x) - T_n^2(x) = (xT_m(x) - T_{m-1}(x))U_{m-1}.$$

Thus to prove the theorem, it remains to show this second identity holds:

$$(T_{m+1}(x) - xT_m(x))U_{m-1}(x) = T_m^2(x) - 1.$$

To prove this, we note that $xT_m(x) - T_{m-1}(x)$ is the weight of all restricted $m + 1$ -tilings that end with a domino or light square. We call these tilings *doubly-restricted*, since they cannot begin or end with a dark square. Let \mathcal{S}_{m+1} denote the set of doubly-restricted $(m + 1)$ -tilings. Again, $U_{m-1}(x)$ counts the weight of all unrestricted $(m - 1)$ -tilings. Next we show a near 1-1 correspondence between elements of $\mathcal{S}_{m+1} \times \mathcal{U}_{m-1}$ and $\mathcal{T}_m \times \mathcal{T}_m$, under tailswapping.

Let (A, B) be in $\mathcal{S}_{m+1} \times \mathcal{U}_{m-1}$. Then position A (length $m + 1$) above B (length $m - 1$) so that B starts once space after A begins and B ends one space before A ends. If possible, tailswapping A and B yields two length m tilings, C and D , so that C is a restricted on the left (the usual restriction) and D is restricted on the right (i.e. its reversal, D^{-1} , is restricted on the left). Then we have $(C, D^{-1}) \in \mathcal{T}_m \times \mathcal{T}_m$.

The only case where tailswapping A and B is impossible is when m is odd and both tilings are all dominoes. The only case where an element $(C, D^{-1}) \in \mathcal{T}_m \times \mathcal{T}_m$ has no preimage under tailswapping is when m is even and C and D are all dominoes.

In the case of m odd,

$$(T_{m+1}(x) - xT_m(x))U_{m-1}(x) - T_m^2(x) = (-1)^{(m+1)/2+(m-1)/2} = -1.$$

If m is even,

$$T_m^2(x) - (T_{m+1}(x) - xT_m(x))U_{m-1}(x) = (-1)^{m/2+m/2} = 1.$$

So in both of these cases we have

$$(T_{m+1}(x) - xT_m(x))U_{m-1}(x) = T_m^2(x) - 1.$$

□

2.2.5 The Composition Identity

The next Chebyshev identity we prove is one of the most fundamental. We call this the composition identity: For $m, n \geq 0$,

$$T_m(T_n(x)) = T_{mn}(x), \tag{2.7}$$

i.e. the composition of the m th and n th Chebyshev polynomials of the first kind is the mn th Chebyshev polynomial of the first kind. Likewise, there is composition identity for the Chebyshev polynomials of the second kind which is similar but slightly less elegant: For $m, n \geq 0$

$$U_{m-1}(T_n(x))U_{n-1}(x) = U_{nm-1}(x). \tag{2.8}$$

We note that these identities are easy to prove using the trigonometric definition of Chebyshev polynomials:

$$\begin{aligned}\cos(m\theta) &= T_m(\cos(\theta)), \\ \frac{\sin(m+1)\theta}{\sin(\theta)} &= U_m(\cos \theta).\end{aligned}$$

for $0 < \theta < 2\pi$, as described in the introduction (Equations 1.1 and 1.2). The composition identity can be derived quickly from the trigonometric definition. Observe that applying the definition to $\cos(mn\theta)$ yields

$$\begin{aligned}T_{mn}(\cos(\theta)) &= \cos(mn\theta) \\ &= T_m(\cos(n\theta)) \\ &= T_m(T_n(\cos \theta)).\end{aligned}$$

Substituting in $x = \cos(\theta)$ gives the composition identity. Deriving the composition identity for $U_{mn-1}(x)$ is equally easy.

Seeing the importance of this identity motivated us to look for combinatorial proof. Such a combinatorial proof would help demonstrate the robustness and usefulness of our combinatorial interpretation of the Chebyshev polynomials. We now give a proof of the composition identity for the Chebyshev polynomials of the first kind.

Theorem 14. *If m, n are nonnegative integers, then*

$$T_m(T_n(x)) = T_{mn}(x). \quad (2.9)$$

To prove this identity we show a bijection between the weights of length mn tilings and the weights of length m metatilings, in which each square has a weight given by a length n minitiling. To elucidate this idea we define what we mean by *metatiling* and *minitiling*.

Definition 9. *A length m metatiling of length n minitilings is a tiling of a length m strip using light dominoes of weight -1 and light and dark squares, each having weights corresponding to the weighted n -tilings. We call the set of all such tilings $\mathcal{U}_m(\mathcal{U}_n)$. The set of metatilings that have the restriction that the first metatiling cannot be a dark square and that no minitiling can begin with a dark square is the set denoted $\mathcal{T}_m(\mathcal{T}_n)$.*

The intuitive idea behind restricted metatilings is that they count tilings corresponding to the combinatorial interpretation of $T_m(T_n(x))$. (Likewise, the unrestricted metatilings count tilings corresponding to $U_m(U_n(x))$.) Note

that $T_m(T_n(x))$ is just our ordinary Chebyshev polynomials of the first kind, $T_m(x)$, where the x has been replaced by $T_n(x)$. Since x corresponded to the weight of the squares, $T_m(T_n(x))$ is interpreted as the sum of the weighted tilings of length m where squares are composed of minitilings having weights summing to $T_n(x)$.

For example, $T_3(T_2(x))$ is the sum of length 3 metatilings in which squares have weights corresponding to length 2 minitilings. Figure 2.4 shows three examples of such tilings. Now we are ready to prove this identity.

Proof. **Case: n odd**

We first prove the result for the case when n is odd. To do this we show a weight preserving bijection between the tilings in \mathcal{T}_{mn} and the metatilings in $\mathcal{T}_m(\mathcal{T}_n)$

Consider a tiling $\sigma \in \mathcal{T}_{mn}$. Write σ as a tiling of m length n rows stacked on top of each other (called an $m \times n$ board), where the first row consists of the cells 1 through n , the second row consists of cells $n + 1$ through $2n$, ..., the m th row consists of cells $(m - 1)n + 1$ through mn . If a domino starts in the last cell of row and ends in the first cell of the next row, DO NOT PANIC! We say that such domino is *out of phase*. Figure 2.5 shows an example of how a the length 18 tiling $DababDDDbDabb$ can be turned into a 6×3 board. Notice that the out of phase domino starting on row 3 and ending on row 4 is denoted by the dashed lines.

The overall picture of the bijection is the following: The k th row of the board is a length n tiling that will correspond to the tile in the k th cell of the metatiling and the associated length n minitiling of that cell.

To determine the element of $\mathcal{T}_m(\mathcal{T}_n)$ the bijection sends a particular element of \mathcal{T}_{mn} to, we use the following algorithm, which starts from the first row of the $m \times n$ board and proceed downwards: (see Figures 2.6, 2.7, and 2.8 for examples)

Case 1a: (Row has no out of phase dominoes. Starts a or D .) Suppose the given row (call it row k) does not contain part of an out of phase domino and begins with a light square or a domino. Then, the board is mapped to a metatiling with a light square at cell k whose embedded minitiling is the same as the tiling of row k . For example, in Figure 2.6 we see that the first row (Da) begins with a domino, so the corresponding metatiling has a light square in the first cell with embedded minitiling Da , the same as the first row of the board. The mapping of row 6 in Figure 2.6 is another example of this case.

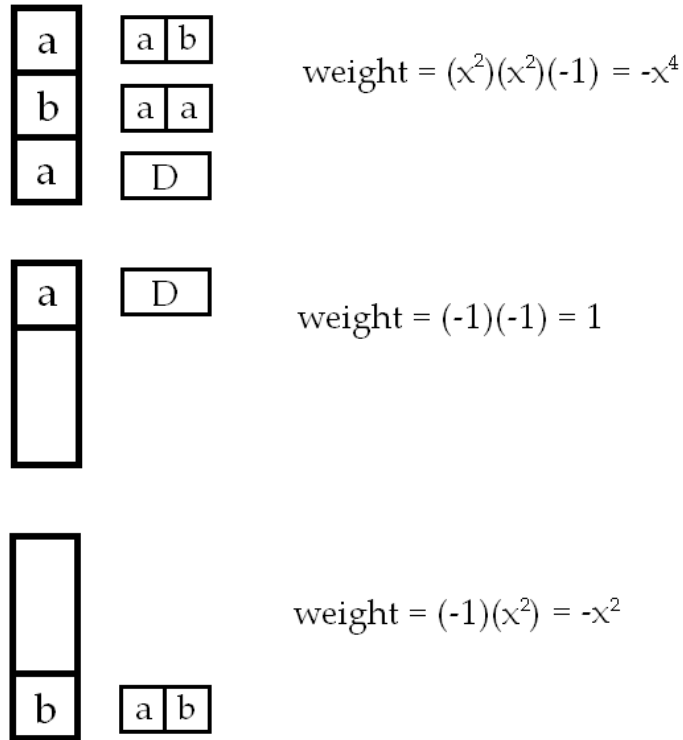


Figure 2.4: Here are three examples of metatilings in the set $\mathcal{T}_3(\mathcal{T}_2)$. Notice that each square in each length 3 metatiling has an embedded length 2 minitiling. The weight of the square in the metatiling is the same as the weight of the embedded minitiling. Dominoes do not have embedded minitilings and always have weight -1.

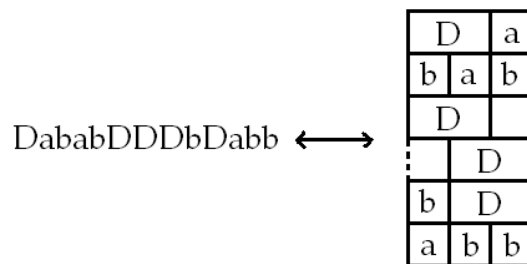


Figure 2.5: This is an example of how to convert an 18-tiling to 6×3 board.

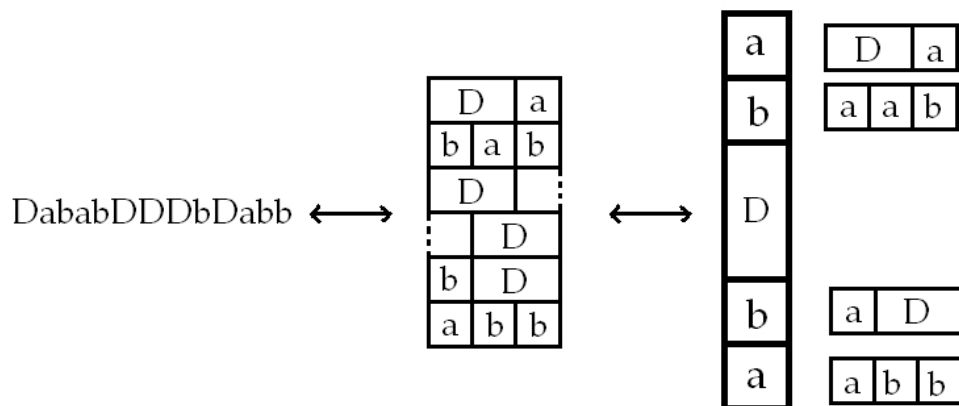


Figure 2.6:

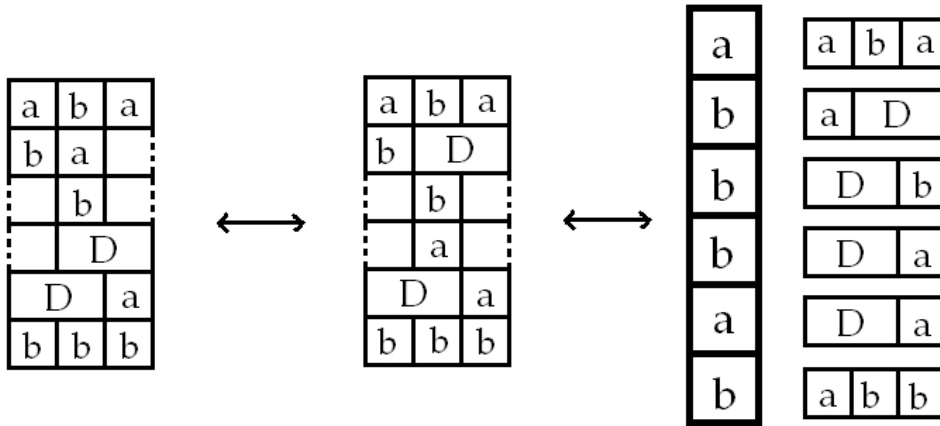


Figure 2.7:

Case 1b: (Row has no out of phase dominoes. Starts *b*.) Suppose row k does not contain part of an out of phase domino and begins with a dark square. Then, the board is mapped to a metatiling with a dark square at cell k whose embedded minitiling is the same as the tiling of row k , except the initial dark square is changed to a light square. (This color swap is made so that the resulting minitiling is an element in \mathcal{T}_n .) For example, in Figure 2.6 we see that the second row (bab) begins with a dark square. Thus the corresponding metatiling has a dark square in the second cell and the embedded minitiling is nearly the same (aab), except the initial square has changed color from dark to light.

Case 2a: (Row ends with out of phase dominoes. Tailswappable.) Suppose row k is the first row to contain part of an out of phase domino. Since it is the first such row, it must have the out of phase domino starting at the last cell. Since row k contains an out of phase domino, it cannot be mapped directly to cell k of the metatiling (and its embedded minitiling). Instead, we must first tailswap row k with the first row after k that does not end with an out of phase domino (call this row j). If rows k and j are tailswappable, tailswap them. For an example of tailswapping see Figure 2.7.

Once this tailswap has occurred, row k no longer contains an out of phase domino. Therefore, apply case 1 to obtain cell k of the corresponding metatiling and its corresponding minitiling.

In contrast, each of the rows $k + 1, k + 2, \dots, j - 1, j$ has part of an out of phase domino in cell 1 and in cell n . Each row is mapped to a dark

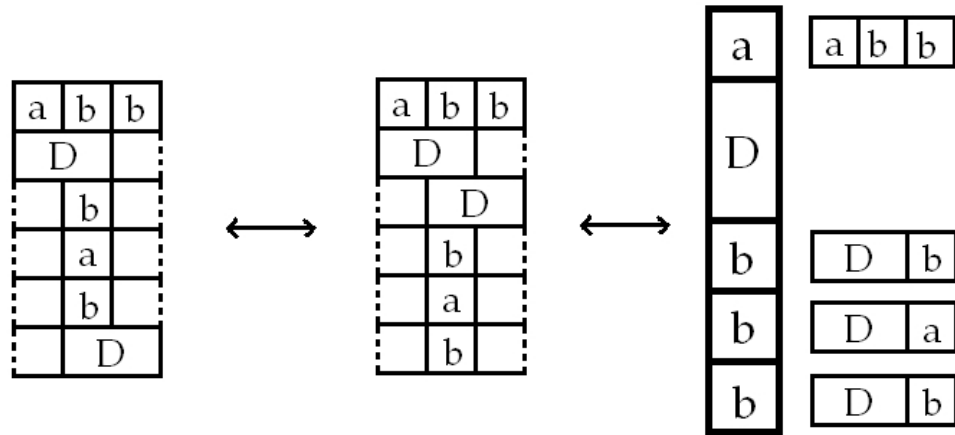


Figure 2.8:

square in the metatiling where the embedded minitiling is the tiling of the row, except the two out of phase domino pieces are put together to form a domino at the beginning.

In Figure 2.7, $k = 2$ and $j = 4$. So rows 2 and 4 are tailswapped. After tailswapping, new row 2 is mapped using case 1(b), after which new rows 3 and 4 are mapped as just described.

Case 2b: (Row ends with out of phase dominoes. NOT tailswappable.) Again, suppose row k is the first row to contain part of an out of phase domino in cell n and that row j is the first row after k that does not end in an out of phase domino. Unlike case 2(a), suppose now that rows k and j cannot be tailswapped. Since n is odd, the only situation where this is possible is when rows j and k contain only dominoes. In this case, insert the tiles in row j between rows k and $k + 1$, effectively shifting rows $k + 1$ through $j - 1$ down one row. The cells k and $k + 1$ of the corresponding metatiling are covered by a domino.

For a trivial example, in Figure 2.6, $k = 3$ and $j = 4$. Since these rows are already next to each other, $k + 1 = j$, so no shifting needs to be done. For a more detailed example, see Figure 2.8. Here $k = 2$ and $j = 6$, so row 6 becomes row 3 and rows 3,4,5 are shifted down to 4,5,6.

Applying the algorithm to each row, starting at row 1 and working down, yields each element in \mathcal{T}_{mn} , thought of as an $m \times n$ board, is sent

to a unique element of $\mathcal{T}_m(\mathcal{T}_n)$. To show that this algorithm produces a bijection is straightforward. However, we will now point out some of the subtleties of the process.

First, notice that every image of a board in \mathcal{T}_{mn} is in fact an element of $\mathcal{T}_m(\mathcal{T}_n)$. Since the first row of the board cannot start with an out of phase domino or dark square, the first tile of the metatiling must be either a light square or a domino, which fits the restriction on the metatilings. Furthermore, any minitiling embedded in any square of the metatiling cannot start with a dark square, since in case 1(b), if row k of the board starts with a dark square, cell k is a dark square, but the color of the first tile in the embedded minitiling is switched from dark to light. So the image of each board satisfies the restrictions of $\mathcal{T}_m(\mathcal{T}_n)$.

Second, the map is surjective. Every element in $\mathcal{T}_m(\mathcal{T}_n)$ has a preimage, which can be found by applying the inverse of this algorithm, starting at row m and working up the rows.

Third, the map is injective. Suppose two $m \times n$ boards have the same image. Since the image is created by working linearly down the rows (or a block of contiguous rows if tailswapping occurs) of the board, the only way two boards could have images with identical metatilings and embedded minitilings is if each of the rows were the same.

Finally, notice that the map preserves weights. Each cell k of a metatiling tiled by a square is the image of a row of the corresponding board with the same tiling as the embedded minitiling. The color of an initial square might have changed, but the weight is preserved. Each domino (weight -1) covering a pair of cells $k, k + 1$ in the metatiling is the image of two rows of the corresponding board that are covered by all dominoes. The weight of those two rows is $(-1)^{2m/2} = (-1)^m = -1$, since we are considering the case where m is odd. Thus the maps preserves weights in this case as well.

Case: n even

For the case where n is even, we use the exact same algorithm to map elements from \mathcal{T}_{mn} to $\mathcal{T}_m(\mathcal{T}_n)$. However, notice that because n is even tailswapping is always possible, eliminating the need for case 2(b). It follows that no image of an $m \times n$ board contains a domino. Furthermore, no image of an $m \times n$ board contains the metatiling ab with each embedded minitiling containing $n/2$ dominoes ($D^{n/2}$). So the map is not surjective (though still injective). To remedy this problem, we show that the sum of the weights of the elements in $\mathcal{T}_m(\mathcal{T}_n)$ not hit by the map is 0.

Suppose $\sigma \in \mathcal{T}_m(\mathcal{T}_n)$ has a metatiling containing a domino or sequence ab with $D^{n/2}$ for each minitiling. If the first such occurrence is a domino,

let $f(\sigma)$ be the tiling where domino is replaced by the sequence ab with $D^{n/2}$ for each minitiling. On the other hand, if the first such occurrence is ab with $D^{n/2}$ for each minitiling, let $f(\sigma)$ be the tiling where the ab has been replaced by a domino. Hence $w(f(\sigma)) = -w(\sigma)$. By pairing up each element with its image under f , we see that combined weight of all such tilings is zero:

$$w(\sigma) + w(f(\sigma)) = w(\sigma) - w(\sigma) = 0.$$

Hence, in the n even case, while our weight-preserving map is no longer a bijection between \mathcal{T}_{mn} and all of $\mathcal{T}_m(\mathcal{T}_n)$, we do have a weight preserving bijection from \mathcal{T}_{mn} to subset of $\mathcal{T}_m(\mathcal{T}_n)$, where the sum of the weights of elements not hit by the bijection is 0.

Therefore,

$$T_m(T_n(x)) = T_{mn}(x).$$

□

Next we will prove the related composition identity for the Chebyshev polynomials of the second kind. The proof uses the same weight preserving bijection as the previous proof, but with a few minor changes. First, we make two quick definitions that will be useful in the next proof.

Definition 10. *A row of a board is closed on the left if its first cell does not contain half of an out of phase domino. Likewise, a row is closed on the right if its last cell does not contain half of an out of phase domino.*

Definition 11. *A row of a board is open on the left if its first cell contains half of an out of phase domino. Similarly, a row is open on the right if its last cell contains half of an out of phase domino.*

Theorem 15. *If n, m are nonnegative integers, then*

$$U_{m-1}(T_n(x))U_{n-1}(x) = U_{mn-1}(x). \quad (2.10)$$

Proof. Consider the tilings in the set \mathcal{U}_{mn-1} . Write each tiling as an $m \times n$ board with the first cell removed, referred to as a *notched board*. Our goal is to convert each notched board into an unrestricted $n - 1$ regular tiling and an unrestricted length $m - 1$ metatiling with restricted length n minitilings. Hence we want a weight preserving bijection taking \mathcal{U}_{mn-1} to

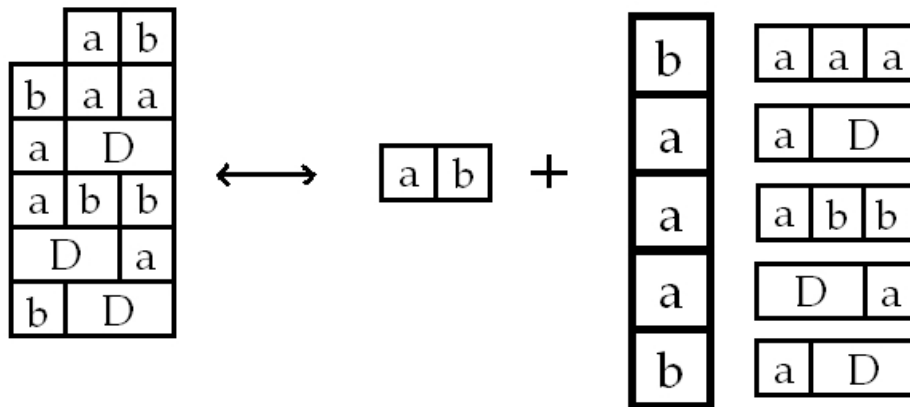


Figure 2.9:

$\mathcal{U}_{n-1} \times \mathcal{U}_{m-1}(\mathcal{T}_n)$. The overall idea is that the first row of the notched board corresponds to the unrestricted $(n - 1)$ -tiling in \mathcal{U}_{n-1} . The remaining $m - 1$ by n board corresponds to the unrestricted length $m - 1$ metatiling with restricted length n minitilings in $\mathcal{U}_{m-1}(\mathcal{T}_n)$.

Suppose that row 1 of the notched board is closed on the right. Then row 1, of length $n - 1$, is mapped directly to the unrestricted $n - 1$ tiling. Then we are left with an $m - 1$ by n board that is closed at the left of its first row. See Figure 2.9 for an example of this case.

On the other hand, suppose that row 1 of the notched board is open on the right. To be able to map row 1 to the unrestricted $(n - 1)$ -tiling, we need it to be closed on the right. So we will tailswap it with the first available row that is closed on the right. Since cell 1 has been removed, the first row is always breakable after cell 1. Hence, we are guaranteed that we can tailswap row 1 with the first row that is closed on the right. Once the tailswap has been performed, we map the new row 1 (now closed on the right) directly to the unrestricted $n - 1$ tiling. We are then left with an $m - 1$ by n board that is open on the left of its first row. See Figure 2.10 for an example of this case.

Case: n odd

Now, we convert the $m - 1$ by n board, in $\mathcal{U}_{(m-1)n}$, to an unrestricted $m - 1$ metatiling with embedded restricted n tilings, in $\mathcal{U}_{n-1}(\mathcal{T}_n)$. We do this just as in the n odd case of the previous proof with one exception. Note that because our $m - 1$ by n board can be open on the left of its first row

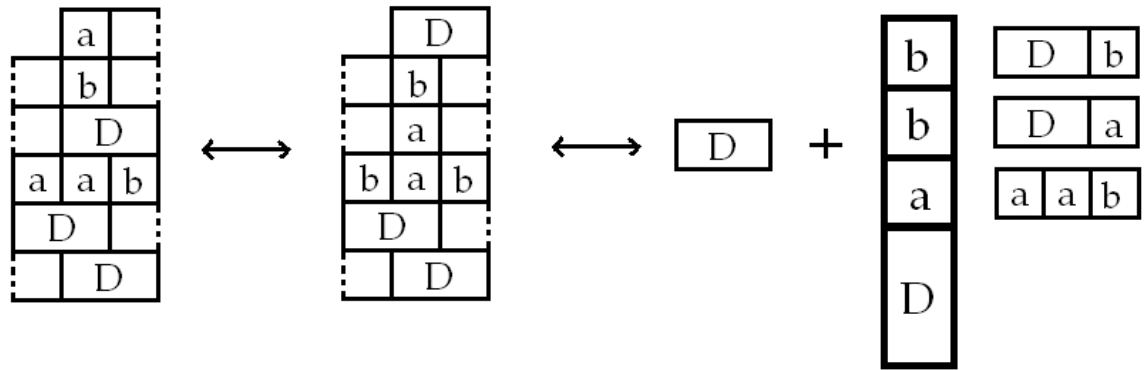


Figure 2.10:

(which could not happen in the previous proof), the corresponding $m - 1$ metatilings are no longer restricted. For instance, in Figure 2.2.5, the first row of the 5 by 3 board is open on the left, and therefore the first cell in the length 5 metatiling is a dark square, which could not have happened in the previous proof. Thus it makes sense that our $m - 1$ by n boards get mapped to *unrestricted* length $m - 1$ metatilings.

Case: n even

We convert the $m - 1$ by n board to an unrestricted $m - 1$ metatiling with embedded restricted n tilings. We do just as in the n even case of the previous proof, again noting that because the first row of the board can be open on the left, our metatilings are unrestricted.

Like the last proof, this mapping is a weight preserving bijection in the case of n odd and a weight preserving near bijection, where the sum of the weights of the elements without preimages in under the map is equal to 0, in the case of n even. Thus we have

$$U_{m-1}(T_n(x))U_{n-1}(x) = U_{mn-1}(x).$$

□

Chapter 3

Conclusion

Over the course of this project, we have found the weighted tiling interpretation of Chebyshev polynomials useful in coming up with combinatorial proofs of numerous Chebyshev identities. Not only have we come up with a combinatorial proof for each identity we examined, these combinatorial proofs give us insight into what each identity means. In addition, the fact that we can prove identities that do not appear combinatorial on the surface, like the composition identity $T_m(T_n(x)) = T_{mn}(x)$, which looks like a statement about composing two continuous functions, further demonstrates the robustness of this combinatorial interpretation.

There were two areas that we only briefly worked on during this project that deserve future attention. The first is an alternative tiling interpretation of the Chebyshev polynomials. In this paper, we proved that $U_n(x)$ counts the total weight of all weighted n -tilings using light squares of weight x , dark squares of weight x , and dominoes of weight -1 . We can also show that $U_n(x)$ counts the total *number* of all n -tilings using only light and dark squares (no dominoes), where there are x colors of light squares and x colors of dark squares, with the restriction that a light square of the first color cannot be followed by a dark square of the last color. This tiling interpretation is equivalent to the previous interpretation. We believe that nearly all of the proofs in this paper can be proved using this alternative tiling interpretation. However, this is a hypothesis we would like to verify in future work.

Another area that we only briefly addressed during this project is the relationship between Chebyshev identities of the first and second kinds and Fibonacci and Lucas identities. R. G. Buschman demonstrates the following beautiful way of converting between Chebyshev polynomials and

Fibonacci and Lucas numbers in ?:

$$\begin{aligned}T_n(i/2) &= i^n L_n / 2 \\U_n(i/2) &= i^n F_{n+1},\end{aligned}$$

where $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$, and $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$, for $n \geq 2$. Using these equations, we can convert Chebyshev identities into Fibonacci/Lucas identities and vice versa. One thing we would like to explore, given more time, is the relationship between the combinatorial proofs of Fibonacci/Lucas identities to the combinatorial proofs in this project. We would expect to find similar techniques employed in the two proofs, such as tailswapping.

Overall, it has been exciting to see how many different Chebyshev identities can be explained combinatorially using our tiling interpretation.

Appendix A

List of Identities

Here is a list of the most important identities that we have proved in order of appearance in this paper. All of these identities can be found in Rivlin (1990) except for the determinant identities, which come from Weisstein (2007).

Theorem 3: If $n \geq 1$,

$$T_n(x) = U_n(x) - xU_{n-1}(x).$$

Theorem 4: If $n \geq 2$,

$$2T_n(x) = U_n(x) - U_{n-2}(x).$$

Theorem 5: If n is a nonnegative integer,

$$U_n(x) = \sum_{j=0}^n x^j T_{n-j}(x).$$

Theorem 6: If $n \geq 2$,

$$T_n(x) = \sum_{j=1}^n x^j T_{n-j}(x) - U_{n-2}(x).$$

Theorem 7: For $n \geq 1$,

$$T_n(x) = \det \begin{bmatrix} x & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2x & 1 & 0 & \ddots & 0 & 0 \\ 0 & 1 & 2x & 1 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 2x & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2x \end{bmatrix},$$

where the matrix is $n \times n$.

Theorem 8: For $n \geq 1$,

$$U_n(x) = \det \begin{bmatrix} 2x & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2x & 1 & 0 & \ddots & 0 & 0 \\ 0 & 1 & 2x & 1 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 2x & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2x \end{bmatrix},$$

where the matrix is $n \times n$.

Theorem 9: If $n \geq 1$,

$$T_n(x) - T_{n-2}(x) = 2(x^2 - 1)U_{n-2}(x).$$

Theorem 11: If m, n are nonnegative integers and $n \geq m$ then

$$2T_n(x)T_m(x) = T_{n+m}(x) + T_{n-m}(x).$$

Theorem 12: If n is a nonnegative integer,

$$(1+x)U_n(x) = 1 + T_{n+1}(x) + \sum_{i=1}^n 2T_i(x).$$

Theorem 13 Let nonnegative integers n, m , such that $n \geq m$. Then

$$T_n^2(x) + T_m^2(x) = 1 + T_{n+m}(x)T_{n-m}(x).$$

Theorem 14: If m, n are nonnegative integers,

$$T_m(T_n(x)) = T_{mn}(x).$$

Theorem 15: If n, m positive integers,

$$U_{nm-1}(x) = U_{m-1}(T_n(x))U_{n-1}(x).$$

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