A Discrete Fourier Transform for the Symmetric Group

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Linear Algebra

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Difficulty of applying T depends on a basis:

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix} \begin{pmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{pmatrix}$$

What we have to do under a very bad basis.

Good news

V decomposes into three T-invariant spaces, W_1, W_2, W_3, with $dim(W_1) = 1, dim(W_2) = 2, dim(W_3) = 2.$

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Let B be a basis that respect this decomposition:

$$B = \{ \underbrace{b_1}_{\text{spans } W_1}, \underbrace{b_2, b_3}_{\text{spans } W_2}, \underbrace{b_4, b_5}_{\text{spans } W_3} \}$$

Applying T in basis B

Under this basis...

$$\begin{pmatrix} \bullet & & & & \\ & \bullet & \bullet & & \\ & \bullet & \bullet & & \\ & & & \bullet & \bullet \\ & & & \bullet & \bullet \end{pmatrix} \begin{pmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{pmatrix} \quad \begin{array}{l} \left.\vphantom{\bullet}\right\} W_1 \\ \left.\begin{array}{c} \vphantom{\bullet} \\ \vphantom{\bullet} \end{array}\right\} W_2 \\ \left.\begin{array}{c} \vphantom{\bullet} \\ \vphantom{\bullet} \end{array}\right\} W_3 \end{array}$$

Figure: $[T]_B[v]_B$.

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We want to use the basis that respects the decomposition of V into T-irreducibles.

$\mathbb{C}G$-module

Group algebra $\mathbb{C}G$

Complex vector space whose basis is the set of group elements of G:

Example : An element of $\mathbb{C}S_3$

$$1(1) + 2(12) + 8(123) + (7+i)(132)$$

In the basis $\{(1), (12), (23), (123), (132), (13)\}$ this can be written as

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 8 \\ 7+i \\ 0 \end{bmatrix}.$$

Group algebra $\mathbb{C}G$

$\mathbb{C}G$ acts on $\mathbb{C}G$ using the group multiplication:

Example in $\mathbb{C}S_2$

$(5(1)+6(12))\bullet(2(1)+3(12)) =$

$$10(1)(1)+15(1)(12)+12(12)(1)+18(12)(12) = 28(1)+27(12)$$

Action of $\mathbb{C}G$ is a linear transformation on $\mathbb{C}G$.

$$\left(5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+6\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ 15 \end{pmatrix}+\begin{pmatrix} 18 \\ 12 \end{pmatrix} = \begin{pmatrix} 28 \\ 27 \end{pmatrix}.$$

As we saw, $\mathbb{C}G$'s action (either left or right) in the vector space $\mathbb{C}G$ can be written as matrix!!! (group representation)

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Such basis is called

a Fourier basis.

Fast Fourier Transform

Keys for the strategy

- Changing the basis in steps via several intermediate bases
- Changing the basis to the Fourier basis amounts to projecting the entire space into many 1 dimensional spaces, each of which is spanned by a Fourier basis vector

Magic with $\mathbb{C}S_3$

$\mathbb{C}S_3$'s decomposition into left $\mathbb{C}S_3$-irreducibles:

$$\mathbb{C}S_3 \cong L_1 \oplus L_2 \oplus L_2' \oplus L_3.$$

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What we saw was same as:

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$\mathbb{C}G$ viewed in the standard basis $\longrightarrow$ time domain
$\mathbb{C}G$ viewed in the matrix basis (a Fourier basis) $\longrightarrow$ frequency domain

Viewing $\mathbb{C}S_3$ through the eyes of $\mathbb{C}S_2$: left action:

A, T are the only types of $\mathbb{C}S_2$-irreducibles.

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$$L_1 \cong A, \qquad L_2 \cong L_2' \cong A \oplus T,$$

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Viewing $\mathbb{C}S_3$ through the eyes of $\mathbb{C}S_2$: right action :

$$\begin{pmatrix} \bullet & & & \\ & \bullet & \bullet & \\ & \bullet & \bullet & \\ & & & \bullet \end{pmatrix} \cong \begin{pmatrix} {\color{blue}\bullet} & & & \\ & {\color{blue}\bullet} & {\color{red}\bullet} & \\ & {\color{blue}\bullet} & {\color{red}\bullet} & \\ & & & {\color{red}\bullet} \end{pmatrix}$$

Viewing $\mathbb{C}S_3$ through the eyes of $\mathbb{C}S_2$: right and left action :

$$\begin{pmatrix} \bullet & & & \\ & \bullet & \bullet & \\ & \bullet & \bullet & \\ & & & \bullet \end{pmatrix} \cong \begin{pmatrix} \color{blue}\bullet & & & \\ & \color{blue}\bullet & \color{yellow}\bullet & \\ & \color{cyan}\bullet & \color{red}\bullet & \\ & & & \color{red}\bullet \end{pmatrix}$$

Decimation in the frequency domain

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- applied from right, distinguishes differently acting right $\mathbb{C}S_k$ irreducibles

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$\mathbb{C}S_n$ decomposes into double coset spaces $\mathbb{C}S_k g S_h$ for each k, $h \leq n$

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Example:

$$\mathbb{C}S_3 = \mathbb{C}S_2(1)S_2 \oplus \mathbb{C}S_2(23)S_2$$

DCF

DCF

The spaces resulting from

- decimation of $\mathbb{C}S_n$ in the frequency domain followed by
- decimation of $\mathbb{C}S_n$ in the time domain

are called double coset frequencies.

Orrison Decimation in Frequency (ODIF) FFT implements the Fourier transform in a series of changes of basis within DCFs.

Bounds of ODIF for the $\mathbb{C}S_n$

n	ODIF full bound	Jameson-Froehle	Clausen DIT	n^2
3	5.3	5.3	13	9
4	10.5	9.7	31	16
5	18.6	15.5	60	25
6	31.4	22.7	102.5	36
7	52.6	32.9	161	49
8	89.9	—	238	64
9	158.8	—	336	81
10	290.5	—	457.5	100
11	547.8	—	605	121
12	1060.7	—	781	144
13	2107.0	—	988	169

Figure: Comparison of (operation count / $n!$).

Other miscellaneous results

- Theorem: Clausen's decimation in time algorithm and ODIF are module theoretically equivalent.
- Two double coset spaces of $\mathbb{C}S_k g S_h$ of the same size are isomorphic as $(\mathbb{C}S_k, \mathbb{C}S_h)$ bimodules.
- Theorem: ODIF's intermediate bases can be constructed with tensor products of the Fourier basis of the smaller symmetric groups.

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