Math 197: Senior Thesis

Traveling Waves in Viscoelastic Fluids

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Introduction

Viscoelastic fluids are a broad class of fluids that exhibit both viscous and elastic properties. Fluid elasticity is the measure of a fluid's tendency to return to its original shape in the absence of external forces, and fluid viscosity is the measure of its resistance to flow. Common examples of viscoelastic fluids include bio fluids, gels, egg white, and corn starch in water. Unlike Newtonian fluids, which have a constant viscosity, viscoelastic fluids have a viscosity that depends on the amount of stress being applied to the fluid.

Our model is a generalization of incompressible fluid flow in one dimension in that it accounts for variations in fluid stress:

$$u_t + uu_x = \sigma_x, \qquad (1)$$

where u denotes fluid velocity and σ denotes fluid stress. To derive a governing equation for stress, we assume that each fluid particle can be approximated as a Maxwell element, a simple mechanical analog for a viscoelastic fluid particle. This assumption yields

$$\lambda(\sigma_t + u\sigma_x - \sigma u_x) = \mu u - \sigma, \qquad (2)$$

where μ is the polymeric viscosity of the fluid and λ is the relaxation time.

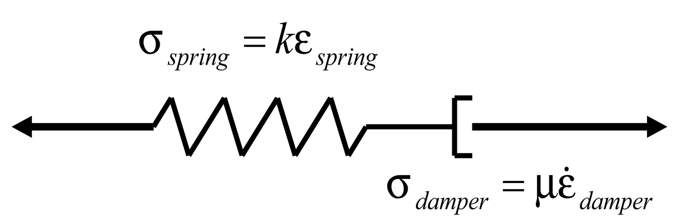


Figure 1: Maxwell Element: A spring and damper connected in series. The variable ε is used to denote strain.

We study a non-dimensionalized version of the (1)-(2) system with asymptotically constant boundary conditions. That is,

$$u_t + uu_x = \sigma_x$$
$$\sigma_t + u\sigma_x = (\sigma + A)u_x - \sigma,$$

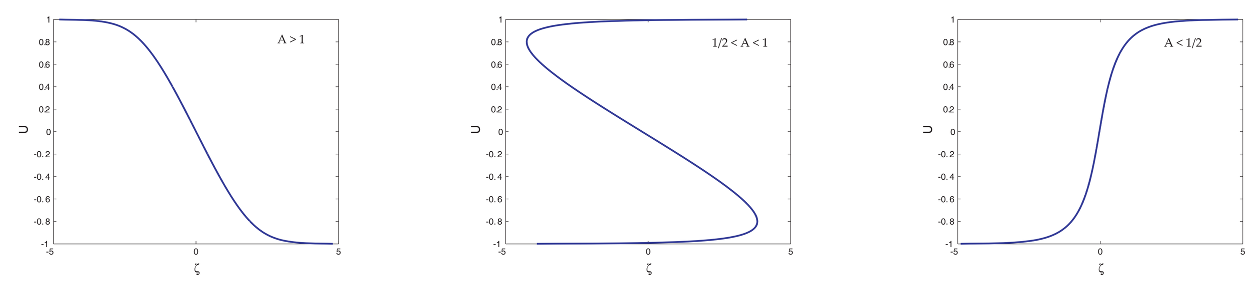
subject to

$$\lim_{x\to-\infty} u(x,t) = 1 \quad \lim_{x\to\infty} u(x,t) = -1 \quad \lim_{|x|\to\infty} \sigma(x,t) = 0.$$

Traveling Wave Solutions

Traveling wave solutions have the form $u(x,t) = U(\zeta)$ and $\sigma(x,t) = S(\zeta)$, where $\zeta = x - ct$. On substituting these function into the PDE system, we find that the following conditions must hold:

$$c = 0, \quad S = \frac{U^2 - 1}{2}, \quad U' = \frac{1 - U^2}{U^2 + 1 - 2A}.$$



(a) $A \in (1,\infty)$). (b) $A \in (1/2,1)$. (c) $A \in (0,1/2)$.

Figure 2: Implicit traveling wave solutions for various values of A.

When $A \in (1,\infty)$, the solution is analytical and satisfies the boundary conditions. When $A \in (1/2,1)$, the solution is multi-valued. When $A \in (0,1/2)$, the solution is analytical but the boundary conditions are reversed. From this analysis, it is clear that traveling wave solutions of the PDE system must be non-classical when $A \in (0,1)$.

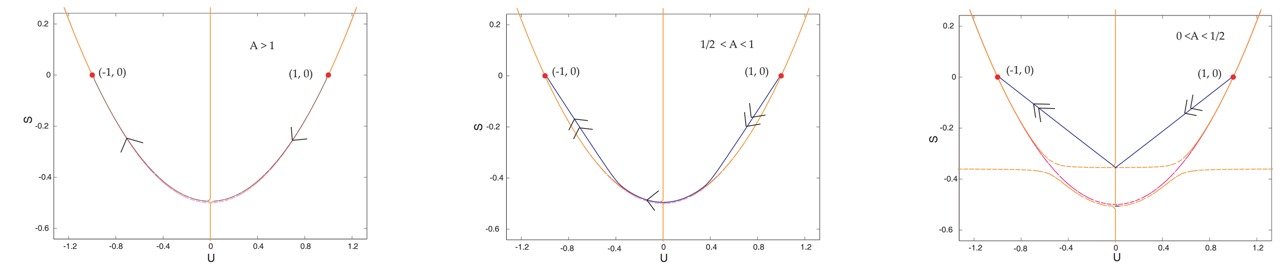
Regularization

To better understand the behavior of the PDE system, when $A \in (0,1)$, we add a viscous term. This new term prevents the formation of shocks. The viscous PDE system is

$$u_t + uu_x = \sigma + \varepsilon u_{xx}$$
$$\sigma_t + u\sigma_x = (\sigma + A)u_x - \sigma,$$

subject to the same boundary conditions as before. As $\varepsilon \to 0$, the solutions of the viscous PDE system look similar to those of the original. Traveling wave solutions to the viscous system must satisfy

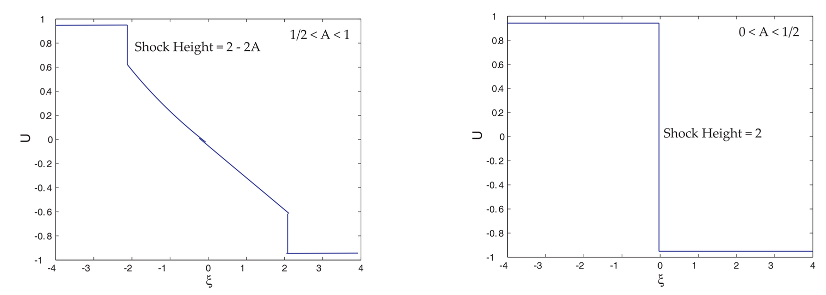
$$U' = \frac{U^2 - 2S - 1}{2\varepsilon}$$
$$S' = \frac{(S+A)(U^2 - 2S - 1) - 2\varepsilon S}{2\varepsilon U}$$



(a) $A \in (1,\infty)$. (b) $A \in (1/2,1)$. (c) $A \in (0,1/2)$.

Figure 3: Solutions in the phase plane for the viscous ODE system.

There are three qualitatively different classes of traveling wave solutions in the viscous system. For $A \in (1,\infty)$, the solution approaches the same form that was observed with the previous system. The pieces of the trajectories that travel in a straight line, as seen in Figure 3b and 3c, correspond to shocks. For $A \in (1/2,1)$, the traveling wave solution converges to a function containing two shocks of equal height as $\varepsilon \to 0$. For $A \in (0,1/2)$, the traveling wave solution converges to a simple shock solution as $\varepsilon \to 0$. We are able to analytically determine the height of the shocks in each case.



(a) $A \in (1/2,1)$. (b) $A \in (0,1/2)$.

Figure 4: Traveling wave solutions for the viscous PDE system as $\varepsilon \to 0$.

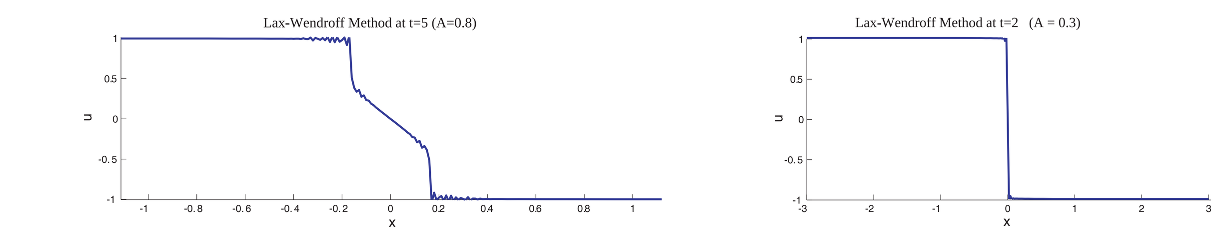
Computer Simulations

To assist with the analysis of our PDE model, we developed a graphical user interface program, called VISCO, that simulates solutions. The numerical algorithm used by VISCO is a fractional-step Lax-Wendroff method [1]. We approximate a small step forward in time using a Taylor approximation.

$$u(x,t_0+\Delta t) = u(x,t_0) + \Delta t u_t(x,t_0) + \frac{\Delta t^2}{2}u_{tt}(x,t_0) + O(\Delta t^3)$$

$$\sigma(x,t_0+\Delta t) = \sigma(x,t_0) + \Delta t\sigma_t(x,t_0) + \frac{\Delta t^2}{2}\sigma_{tt}(x,t_0) + O(\Delta t^3)$$

Using the PDE system, all t-derivatives in the above equations can be expressed in terms of x-derives. The x-derivatives are approximated using centered first order finite difference approximations. This gives an explicit second order numerical method. Repeated simulations showed a tendency for all initial data to converge to travelling wave solutions. Figure 5 shows the numerically stable solutions for two different values of A. These solutions are consistent with our analysis of the viscous PDE system.



(a) $A \in (1/2,1)$. (b) $A \in (0,1/2)$.

Figure 5: Globally stable solutions of our numerical method.

Acknowledgments

I would like to thank my thesis advisor, Darryl Yong, for his positive attitude, patience, and devotion to my personal development through this research experience. Professor Jon Jacobsen, for introducing me to this project and guiding my research in the summer of 2006. Bob Guy, for helping me to understand many difficult aspects of this thesis project. The Baker Foundation, for generously funding my work.

References

[1] R. LeVeque, *Finite Volume Methods for Hyperbolic Problems*, Cambridge University Press, 2002.

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