

# Functions of the Binomial Coefficient

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# Abstract

The well-known binomial coefficient is the building block of Pascal's triangle. We explore the relationship between functions of the binomial coefficient and Pascal's triangle, providing proofs of connections between Catalan numbers, determinants, non-intersecting paths, and Baxter permutations.



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# Chapter 1

## Introduction

### 1.1 The Binomial Coefficient

The binomial coefficient, one of the most well-known mathematical objects, is defined as

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1}.$$

Combinatorially, the binomial coefficient counts the number of subsets of size  $k$  from a size  $n$  set. More well known is the fact that the binomial coefficient generates Pascal's triangle.

$$\begin{array}{ccccccc} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ 1 & 3 & 3 & 1 & & & \\ 1 & 4 & 6 & 4 & 1 & & \\ 1 & 5 & 10 & 10 & 5 & 1 & \\ \vdots & & & & & & \ddots \end{array}$$

where  $\binom{n}{k}$  gives the term in the  $n^{\text{th}}$  row and  $k^{\text{th}}$  column. Though relatively easy to construct, Pascal's Triangle has a myriad of interesting connections to other mathematical quantities. For example, summing across a row yields

$$\sum_{k=0}^n \binom{n}{k} = 2^n.$$

Or, summing diagonally yields

$$\sum_{k \geq 0} \binom{n-k}{k} = F_{n+1}$$

where  $F_n$  is the  $n^{\text{th}}$  Fibonacci number.

## 1.2 Functions of the Binomial Coefficient

Since the binomial coefficient and Pascal's Triangle appear to have so many connections to other numerical quantities, we explore the effect of replacing each factor  $i$  in the binomial coefficient with some function  $f(i)$ . For example, the  $n^{\text{th}}$  triangular number is defined as the sum of the first  $n$  positive integers. So, if we let  $f(i) = t_i$ , we have

$$\left[ \begin{matrix} n \\ k \end{matrix} \right] = \frac{t_n t_{n-1} \cdots t_{n-k+1}}{t_k \cdots t_1}$$

which we shall call the triangular coefficient. Or, letting  $f(i) = \binom{i+2}{3}$  we have

$$\left[ \begin{matrix} n \\ k \end{matrix} \right]_3 = \frac{\binom{n+2}{3} \binom{n+1}{3} \cdots \binom{n-k+3}{3}}{\binom{k+2}{3} \cdots \binom{3}{3}}$$

which we shall call a Baxter coefficient. We also explore  $f(i) = F_n$ , yielding

$$\left[ \begin{matrix} n \\ k \end{matrix} \right]_F = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_k \cdots F_1}.$$

We examine new such mathematical objects created by choosing various functions to replace each factor of the binomial coefficient. Rather than algebraically grind out various formulae, we provide insightful combinatorial proofs of theorems and identities for these mathematical objects and demonstrate connections to other areas.

## Chapter 2

# The Catalan Numbers

### 2.1 The Catalan Sequence

The Catalan numbers are a sequence of numbers which play an integral role in the understanding of the triangular coefficient, given in the introduction. The Catalan numbers  $1, 1, 2, 5, 14, 42, \dots$  are defined by  $C_0 = 1$  and satisfy the recurrence

$$C_{n+1} = \sum_{k \geq 0} C_k C_{n-k}$$

with the initial condition  $C_0 = 1$ . The  $n^{\text{th}}$  Catalan number can also be expressed directly in terms of the binomial coefficients

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

### 2.2 Combinatorial Interpretations

Despite the slightly unusual recurrence, the Catalan numbers have many simple combinatorial interpretations. One such interpretation describes lattice paths.  $C_n$  counts the number of lattice paths from  $(0, 0)$  to  $(n, n)$  that only move up and right which lie below the line  $y = x$ . For example, Figure 2.1 shows the first several Catalan paths.

To prove this, we demonstrate that such paths satisfy the Catalan recurrence. Let  $P$  be a Catalan path from  $(0, 0)$  to  $(n+1, n+1)$ . Consider the last point at which  $P$  touches the  $y = x$  line, say point  $(k, k)$  where  $0 \leq k \leq n$ . There are  $C_k$  ways to reach  $(k, k)$  from the origin. Also, we know  $P$  never touches the  $y = x$  line again, so we must move from  $(k+1, k)$  to  $(n+1, n)$

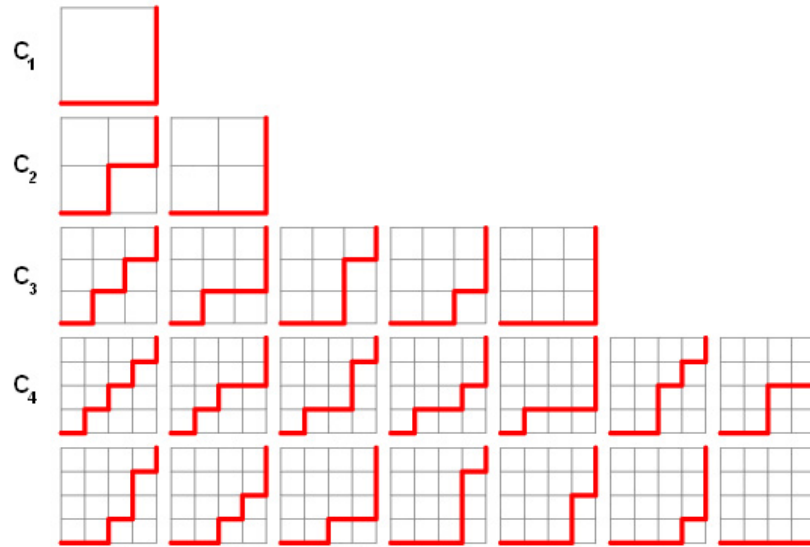


Figure 2.1: The Catalan Paths for  $n=1,2,3$  and 4.

without ever crossing the  $y = x - 1$  line. Here, there are  $C_{n-k}$  ways to move from  $(k + 1, k)$  to  $(n + 1, n)$ . Therefore, there are  $C_k C_{n-k}$  total such paths  $P$ . Summing over all values of  $k$  yields the Catalan Recurrence, as desired. We also define  $C_0$  to be 1, as there is one Catalan path from  $(0, 0)$  to  $(0, 0)$ , namely the empty path.

Catalan numbers have a variety of combinatorial interpretations, which can be found in [6]. Here we consider an interpretation involving polyominoes. We first define a polyomino to be two simultaneous lattice paths of equal length which only move up and right, where both start at  $(0, 0)$  and end at the same point, but never intersect beforehand. Below are examples of polyominoes

Although the polyomino is defined in terms of two paths, we can visually see polyominoes as two dimensional shapes. We define the *width* of a polyomino to be the number of columns in the polyomino and the *height* to be the number of rows. For example, the rightmost polyomino in Figure 2.2 has width 3 and height 4. We can also talk about the height of each column within the polyomino. For instance, the rightmost polyomino has columns of height 2, 3, and 2. Last, we define the *size* of a polyomino to be the length of the paths which forms the polyomino. For example, the first two polyominoes in Figure 2.2 have size 6, as both the red and blue

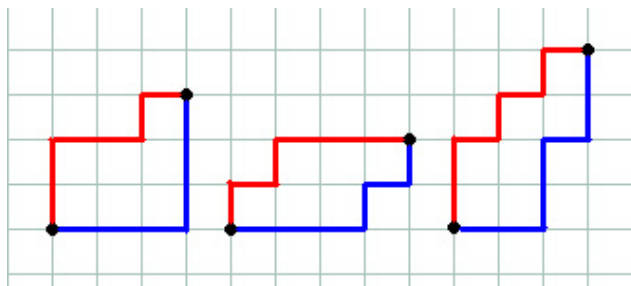


Figure 2.2: Three examples of polyominoes.

paths have length 6. Likewise, the rightmost polyomino has length 7. Let  $P_n$  denote the number of polyominoes of size  $n$ .

We demonstrate that polyominoes satisfy the Catalan recurrence and are, hence, counted by the Catalan sequence. In examining Catalan paths, we looked at the last point where our path touched the  $y = x$  line. Similarly, for polyominoes, we examine the last point at which a polyomino of size  $n + 1$  has two columns that overlap by exactly one square. In the case where no columns overlap by one, such as the leftmost polyomino in Figure 2.2, we say that this last overlap occurs before the first column, at the zeroth break point. Such a polyomino is analogous to a Catalan Path which never touches the  $y = x$  line, and hence the last contact point with  $y = x$  is at  $(0, 0)$ . Suppose this overlap occurs between columns  $i$  and  $i + 1$ . Further suppose that the last move entering this overlap was the  $k^{\text{th}}$  move and, hence, the first move leaving the overlap is the  $(k + 1)^{\text{st}}$  move. We know there are  $P_k$  ways of forming the polyomino to the left of the overlap. To the right of the overlap, we know that there are no more columns which overlap by exactly one square. To create this, we form a polyomino of size  $n - k$ , which can be formed  $P_{n-k}$  ways, and then increase the height of each column by 1, as in Figure 2.3.

By increasing the height of each column by 1, we increase the size of the polyomino by exactly 1. Therefore, we can join the first polyomino, which can be made  $P_k$  ways and the second modified one, which can be made  $P_{n-k}$  ways, to form a polyomino of size  $n + 1$ . Summing over all possible values of  $k$  yields the Catalan Recurrence, as desired. However, we must examine the initial conditions. Note that there are 2 polyominoes of size 3, and 5 polyominoes of size 4. In other words,  $C_{n-1}$  counts the number of polyominoes of size  $n$ , rather than  $C_n$ . Therefore  $P_{n+1} = C_n$ .

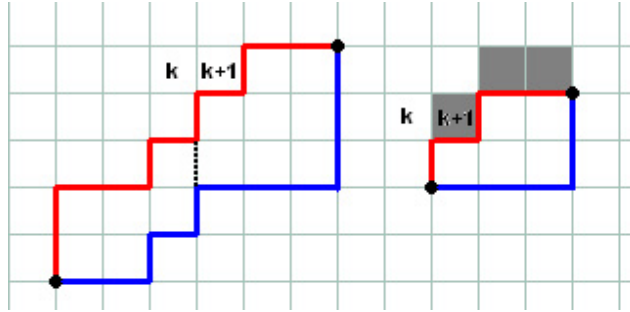


Figure 2.3: The last overlap occurs after move  $k$ . The polyomino to the right of  $k$  can be constructed by forming a polyomino of size  $n - k$  and then increasing each column height by 1, as denoted by the grey boxes.

### 2.3 Higher Dimensional Catalan Numbers

In [3], we see that there are generalizations to higher dimensional Catalan numbers. We call  $C(d, n)$  the number of  $d$ -dimensional lattice paths using the steps  $X_1 := (1, 0, \dots, 0)$ ,  $X_2 := (0, 1, \dots, 0)$ ,  $\dots$ ,  $X_d := (0, 0, \dots, 1)$ , running from  $(0, 0, \dots, 0)$  to  $(n, n, \dots, n)$  and lying in the wedge  $\{(x_1, x_2, \dots, x_d) : 0 \leq x_1 \leq x_2 \leq \dots \leq x_d\}$ . In [3], MacMahon uses generating function arguments to demonstrate that

$$C(d, n) = (dn)! \prod_{i=0}^{d-1} \frac{i!}{(n+i)!}.$$

Therefore, we see that  $C(2, n) = \frac{1}{n+1} \binom{2n}{n}$ . Rather than rely on generating functions, which can often be overly mechanical and non-intuitive, we present a combinatorial proof for the closed formula  $C(2, n)$ . This proof is based on the argument presented in [8].

**Theorem 1.**  $C(2, n) = \frac{1}{n+1} \binom{2n}{n}$

*Proof.* We construct all possible lattice paths from  $(0, 0)$  to  $(n, n)$  only using right and up moves. We can enumerate a path in such a manner by using  $X$  and  $Y$  for right and up moves, respectively. Therefore, one path to  $(3, 3)$  could be enumerated as  $XY Y X X Y$ . Since we are making  $2n$  total moves, we can simply choose where the  $n$   $X$  moves are in a path and the  $Y$  moves are thereby determined. In other words, there are  $\binom{2n}{n}$  total lattice paths to  $(n, n)$ .

We then append a  $Y$  move to the end of each lattice path. As a result, we have a path from  $(0, 0)$  to  $(n, n + 1)$  with  $n$   $X$  moves and  $n + 1$   $Y$  moves which ends in a  $Y$ . We can now create an equivalence class by cycling the enumeration to generate other paths which end in an up move. For example, to the path  $XY Y X X Y$ , we first append a  $Y$ , yielding  $XY Y X X Y \underline{Y}$ . We then cycle this permutation to generate all possible paths ending in an up move (we underline the original last  $Y$  move to help see the cycling):  $XY Y X X Y \underline{Y}$ ,  $\underline{Y} X Y Y X X Y$ ,  $X X Y \underline{Y} X Y Y$ , and  $Y X X Y \underline{Y} X Y$ . These paths together form an equivalence class. Therefore, with  $n + 1$  (or in this example, 4) total  $Y$  moves in each path, there will be a total of  $n + 1$  paths in each equivalence class. Since  $n$  and  $n + 1$  are relatively prime, it is impossible to have duplicate paths in an equivalence class.

Geometrically, each path is a lattice path to  $(n, n + 1)$  that ends in an up move, so we can generate a path to  $(n, n)$  by removing the last move. Also, since there are no lattice points in the triangle bounded by  $y = \frac{n+1}{n}x$ ,  $y = x$  and  $x = (n + 1)$ , we know that the number of paths to  $(n, n)$  which never cross the  $y = x$  line is equal to the number of paths to  $(n, n + 1)$  which never cross the  $y = \frac{n+1}{n}x$  line. Using this fact, given any path  $P$  to  $(n, n + 1)$  we append the identical path  $P'$  to the end, as seen in Figure 2.4.

We then push the  $y = \frac{n+1}{n}x$  line upward until the entire path  $PP'$  is under this parallel line. Naturally, the path will start and end after up moves, as seen in Figure 2.5.

Since  $n$  and  $n + 1$  are relatively prime, we know that these start and endpoints can only be corresponding moves in  $P$  and  $P'$ . Moreover, we see that there is exactly one line placement that will be entirely above  $PP'$ . Therefore, making exactly one of the  $Y$  moves in  $P$  the last up move will result in a path that stays below the  $y = \frac{n+1}{n}x$  line. In other words, in each of our equivalence classes, there is only one element that lies below the  $y = \frac{n+1}{n}x$  line, i.e.,  $\frac{1}{n+1}$  of the total paths. In Figure 2.5, we see that the path in blue is the legitimate Catalan path in the equivalence class. For each of these paths, we simply remove the last up move and we have a Catalan path to  $(n, n)$ . Therefore,  $C_n = \frac{1}{n+1} \binom{2n}{n}$  as desired.  $\square$

We note that rearranging MacMahon's formula yields a combinatorially telling formula

$$C(d, n) = \frac{\binom{dn}{n, n, \dots, n}}{\binom{n+1}{1} \binom{n+2}{2} \cdots \binom{n+d-1}{d-1}}.$$

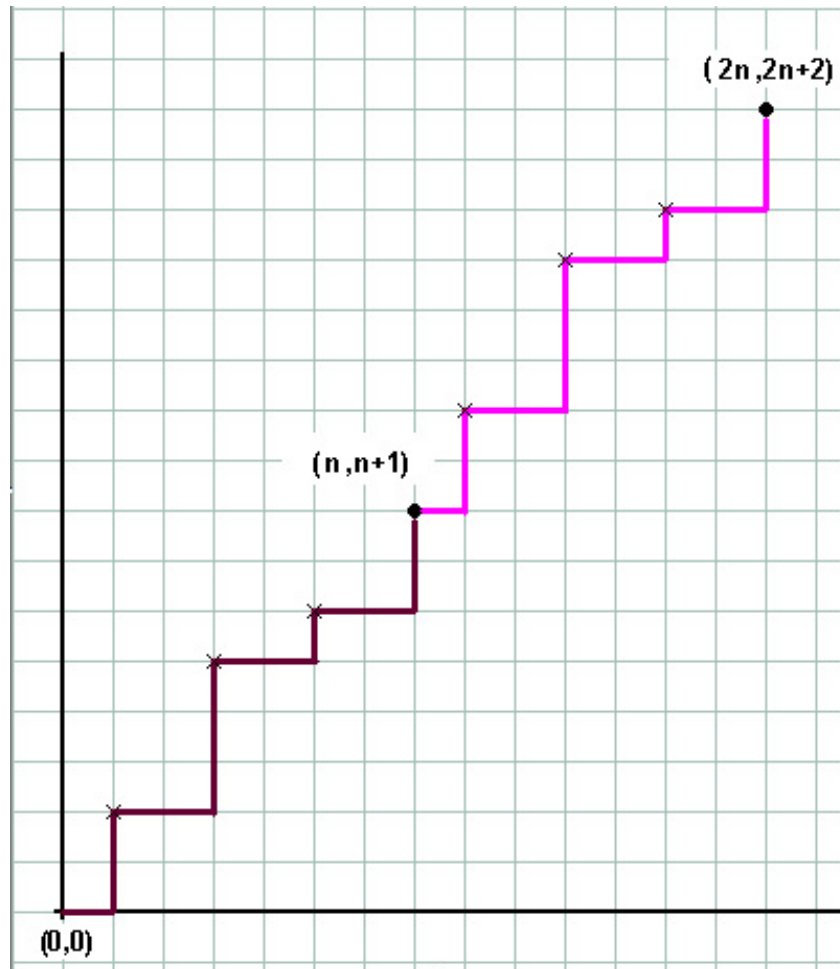


Figure 2.4: The concatenation of  $P$  and  $P'$ .

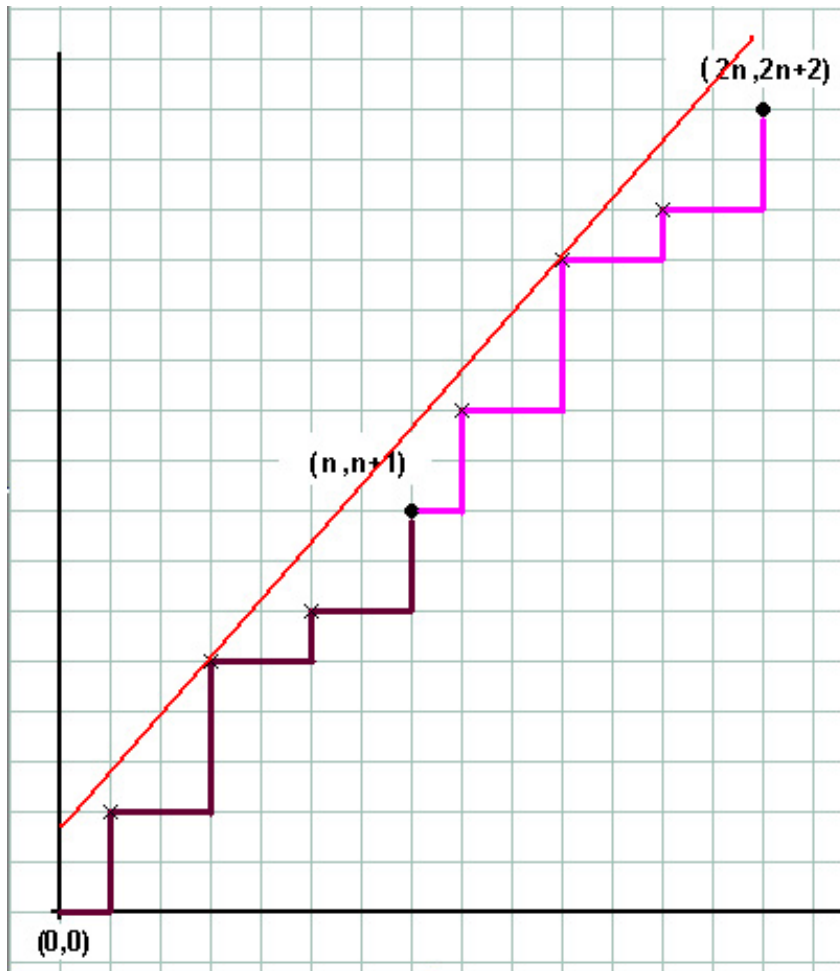


Figure 2.5: After exactly one upmove will  $PP'$  lie under a line parallel to  $y = \frac{n+1}{n}$ .



## Chapter 3

# Baxter Permutations

### 3.1 Introduction to Baxter Permutations

Baxter permutations are a somewhat odd set of restricted permutations of  $[n]$ . Consider a permutation of  $[n]$ , in word form  $\sigma_1\sigma_2\dots\sigma_{n-1}\sigma_n$  where  $\sigma_i$  is the mapping of element  $i$ . A Baxter permutation satisfies the following restrictions: for all indices  $1 \leq i < j < k < l \leq n$ ,

1. If  $\sigma_i + 1 = \sigma_l$  and  $\sigma_j > \sigma_l$ , then  $\sigma_k > \sigma_l$
2. If  $\sigma_l + 1 = \sigma_i$  and  $\sigma_k > \sigma_i$ , then  $\sigma_j > \sigma_i$

Although this might seem somewhat complicated, some examples make Baxter permutations very easy to understand. Let  $n = 5$ . The condition  $\sigma_i + 1 = \sigma_l$  says to choose two consecutive numbers, such as 1 and 2, 2 and 3, 3 and 4, or 4 and 5. For our example, let us choose the two numbers 2 and 3. The restriction  $1 \leq i < j < k < l \leq n$  tells us to look between these consecutive numbers. So, in the permutation 21543, since we have chosen the numbers 2 and 3, we look at the sequence of numbers in between 2 and 3, namely 154. Finally, the condition "if  $\sigma_j > \sigma_l$ , then  $\sigma_k > \sigma_l$ " implies that the sequence 154 must have all numbers below 2 come first, followed by all numbers above 3. So, for example, 21543 satisfies these conditions because, between 2 and 3, we see that the sequence 154 can be divided into 1|54. Clearly, all numbers below 2 come first, followed by all numbers above 3. If the permutation satisfies these conditions for all consecutive pairs of numbers, then we say the permutation is a Baxter permutation. The second restriction does nothing more than consider everything in reverse order. That is, it accounts for when  $i + 1$  comes before  $i$  (such as 3 coming before 2) in a permutation. For example, for the permutation 35412, we see

that, for 3 and 2, the sequence 541 is acceptable because everything above 3 comes first, followed by everything below 2. Therefore, for  $n = 4$ , the only *non*-Baxter permutations are 2413 and 3142.

## 3.2 Background

Baxter permutations originally appeared when trying to prove the “commuting function” conjecture of Dyer [1]. Although Dyer’s conjecture was eventually proven false, Baxter permutations have a more general significance in fixed point analysis:

For a continuous mapping  $h : [0, 1] \rightarrow [0, 1]$  let  $[h] = \{x : h(x) = x\}$  be the set of fixed points of  $h$ . Let  $[h]^* \subset [h]$  denote the set of crossing points of  $h$ . That is  $x^* \in [h]^*$  if and only if  $x^*$  is a limit point of both  $\{x : h(x) < x\}$  and  $\{x : h(x) > x\}$ . For functions  $f, g : [0, 1] \rightarrow [0, 1]$ , if  $x^* \in [g \circ f]$  then

$$(f \circ g)(f(x^*)) = f(g(f(x^*))) = f(x^*).$$

In other words,  $f(x^*) \in [f \circ g]$ . When  $[f \circ g]$  is finite, then  $f$  is a 1-1 map of  $[g \circ f]$  onto  $[f \circ g]$  and induces a permutation of  $\sigma_f$  of  $\{1, 2, \dots, M\}$  onto itself as follows: If we write  $[g \circ f] = \{x_1, \dots, x_M\}$ ,  $[f \circ g] = \{y_1, \dots, y_M\}$ , then for  $i \in \{1, 2, \dots, M\}$ ,

$$\sigma_f(i) = j$$

where  $f(x_i) = y_j$ . So, for example, suppose that we have the sets  $\{x_1, x_2, x_3, x_4\}$  and  $\{y_1, y_2, y_3, y_4\}$  such that  $f(x_1) = y_2$ ,  $f(x_2) = y_3$ ,  $f(x_3) = y_4$  and  $f(x_4) = y_1$ . We can represent this as the permutation  $\sigma_f = 2341$ .

Baxter demonstrated in [1] that when  $|[g \circ f]| = 2n - 1$ , the corresponding permutation is a Baxter permutation.

## 3.3 Various Results

Despite an ugly recurrence relation,  $B(n)$ , the number of Baxter permutations of  $[n]$ , has a rather nice closed formula.

$$B_n = \sum_{k=1}^n \frac{\binom{n+1}{k-1} \binom{n+1}{k} \binom{n+1}{k+1}}{\binom{n+1}{1} \binom{n+1}{2}}.$$

Surprisingly, each term in the sum is an integer. In [4], Mallows gives a combinatorial interpretation of this result, proving that  $k$  is the number of rises in each permutation.

## Chapter 4

# The Triangular Coefficient

### 4.1 The Triangular Coefficient

As mentioned, we obtain the triangular coefficient by replacing each factor  $i$  in the binomial coefficient with  $t_i$ , the  $i^{\text{th}}$  triangular number, yielding

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{t_n t_{n-1} \dots t_{n-k+1}}{t_k \dots t_1}.$$

We immediately see that since  $t_n = \binom{n+1}{2}$  we have that

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{\binom{n+1}{2} \binom{n}{2} \dots \binom{n-k+2}{2}}{\binom{k+1}{2} \binom{k}{2} \dots \binom{3}{2} \binom{2}{2}}.$$

Algebraically, we then see that

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix} &= \frac{\binom{n+1}{2} \binom{n}{2} \dots \binom{n-k+2}{2}}{\binom{k+1}{2} \dots \binom{2}{2}} \\ &= \frac{[(n+1)(n)][(n)(n-1)] \dots [(n-k+2)(n-k+1)]}{[(k+1)(k)] \dots [(3)(2)][(2)(1)]} \\ &= \frac{1}{k+1} \frac{(n+1)(n)(n-1) \dots (n-k+2)}{(k)(k-1) \dots (2)(1)} \frac{(n)(n-1) \dots (n-k+1)}{(k)(k-1) \dots (2)(1)} \\ &= \frac{1}{k+1} \binom{n}{k} \binom{n+1}{k} \end{aligned}$$

which will be a very useful form in future proofs.

Just as the binomial coefficient generates Pascal's Triangle, the triangular coefficient has its own triangle as well.

$$\begin{array}{ccccccc}
 1 & & & & & & \\
 1 & 1 & & & & & \\
 1 & 3 & 1 & & & & \\
 1 & 6 & 6 & 1 & & & \\
 1 & 10 & 20 & 10 & 1 & & \\
 \vdots & & & & & & \ddots
 \end{array}$$

We immediately see several interesting properties. First,  $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}$ . Surprisingly, the triangular coefficient always appears to yield an integer. Most significantly, however, we see that

$$\begin{aligned}
 \sum_{k \geq 0} \begin{bmatrix} n \\ k \end{bmatrix} &= \sum_{k=0}^n \frac{1}{n-k+1} \binom{n}{k} (n+1) \\
 &= \sum_{k=0}^n \frac{1}{n+1} \binom{n+1}{n-k+1} (n+1) \\
 &= \frac{1}{n+1} \binom{(n+1) + (n+1)}{1 + (n+1)} \\
 &= \frac{1}{n+1} \binom{2n+2}{n+2} \\
 &= \frac{1}{n+2} \binom{2n+2}{n+1} \\
 &= C_{n+1}
 \end{aligned}$$

The above identity implies that the triangular coefficients count something, namely that they partition Catalan paths in some way. Although the above properties can be proven algebraically, we provide some insightful combinatorial proofs which demonstrate several interesting connections between the triangular coefficients and the Catalan numbers.

## 4.2 Kinky Proof

As the Catalan path can involve many changes in direction, we choose to define a *kink* in a path as a combinatorial parameter of a Catalan path.

**Definition 1** (kink). A *kink* in a lattice path is a lattice point at which an upward move is followed by a rightward move.

The X in Figure 4.1 denotes the location of a kink.



Figure 4.1: A kink.

Figure 4.2 depicts a change in direction that is *not* a kink.

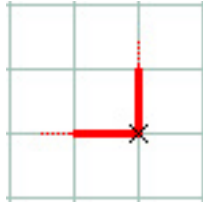


Figure 4.2: A non-kink.

With this definition in mind, we can introduce our first combinatorial interpretation of the triangular coefficient.

**Theorem 2.**  $\begin{bmatrix} n \\ k \end{bmatrix}$  counts the number of Catalan Paths to  $(n+1, n+1)$  with exactly  $k$  kinks.

*Proof.* First, let a lattice path from  $(0,0)$  to  $(n+1, n+2)$  that only makes right and up moves and stays on or below the line  $y = \frac{(n+2)}{(n+1)}x$  be called an  $A_{n+1}$  path. We claim that the number of  $A_{n+1}$  paths is equal to  $C_{n+1}$ . To prove this, we provide a bijection between  $A_{n+1}$  paths and Catalan paths to  $(n+1, n+1)$ . Clearly, every Catalan  $C_{n+1}$  path with an additional upstep at the end is an  $A_{n+1}$  path. Conversely, since there are no lattice points in the triangle bounded by  $y = \frac{(n+2)}{(n+1)}x$ ,  $y = \frac{(n+1)}{(n+1)}x$  and  $x = (n+1)$ , we know that an  $A_{n+1}$  path must lie on or below the  $y = x$  line. Consequently, since

a  $C_{n+1}$  path must have its last move be an up move, an  $A_{n+1}$  path must have its last two moves must be up moves. Therefore, by removing the last upward move from an  $A_{n+1}$  path, we form a Catalan path to  $(n+1, n+1)$ , forming a simple bijection between  $A_{n+1}$  paths and Catalan paths.

We will now construct an unrestricted path from  $(0, 0)$  to  $(n+1, n+2)$  by placing  $k$  kinks. We will force the first move in the path to be a right move and the last move to be an up move. Therefore, we can place a kink at any point  $(x, y)$  where  $x \in \{1, 2, \dots, n\}$  and  $y \in \{1, 2, \dots, n+1\}$ . That is we can place on any lattice point at any of the red dots in Figure 4.3.

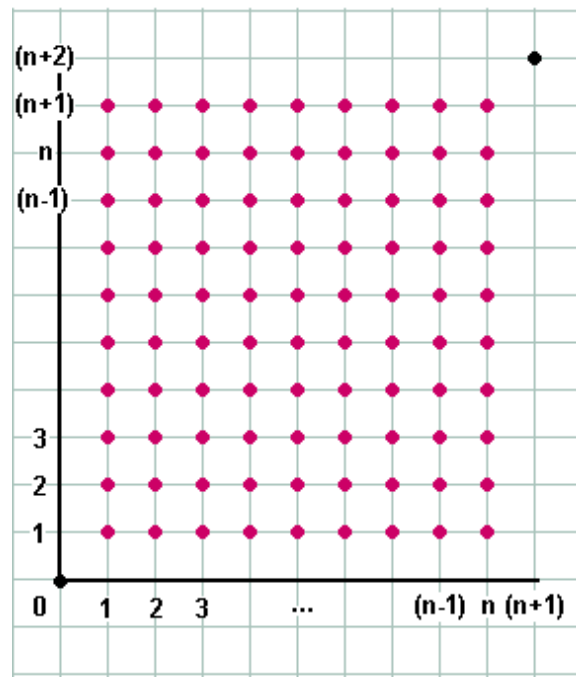


Figure 4.3: The red dots denote lattice points at which we can place kinks.

We also must have the condition that kinks be placed in an increasing order. For example, we could not place kinks as in Figure 4.4.

Therefore, in order to place  $k$  kinks in a legitimate fashion, we choose  $k$   $x$ -coordinates  $1 \leq x_1 < x_2 < \dots < x_k \leq n$  and  $k$   $y$ -coordinates  $1 \leq y_1 < y_2 < \dots < y_k \leq n+1$ . The coordinates of our kinks will then be  $(x_1, y_1) \dots (x_k, y_k)$ . For example, if we chose the  $x$ -coordinates  $(1, 5, 6)$  and the  $y$ -coordinates  $(2, 4, 5)$ , we would place kinks at  $(1, 2)$   $(5, 4)$  and  $(6, 5)$  as in Figure 4.5.

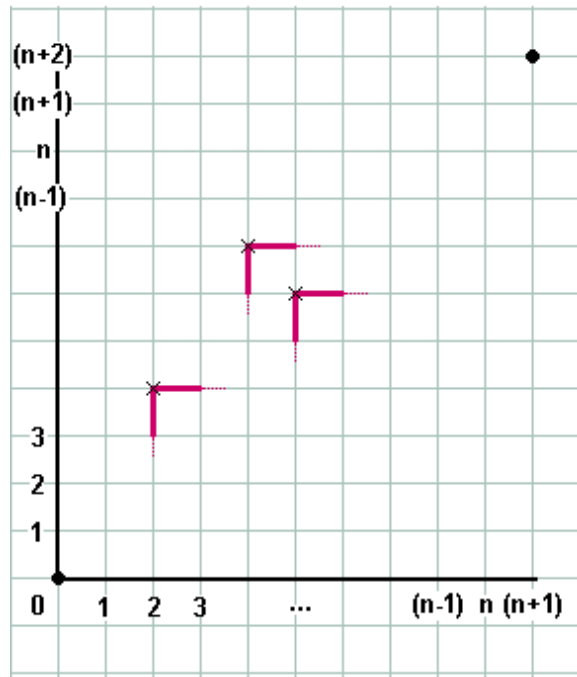


Figure 4.4: An improper arrangement of kinks.

Therefore, there are  $\binom{n+1}{k} \binom{n}{k}$  total paths from  $(0, 0)$  to  $(n + 1, n + 2)$  that can be formed in this manner. We now demonstrate that exactly one out of every  $k + 1$  paths constructed in this fashion yields an  $A_{n+1}$  path. Let  $P$  be a path constructed in this manner with  $k$  kinks. Let us label these kinks as  $a_1, a_2, \dots, a_k$ . We will now generate a valid  $A_{n+1}$  path from  $P$ . Consider  $P'$  as a translation of  $P$  from  $(n + 1, n + 2)$  to  $(2n + 2, 2n + 4)$  with the kinks labeled as  $a'_1, a'_2, \dots, a'_k$ , as shown in Figure 4.6.

We can push the  $y = \frac{(n+2)}{(n+1)}x$  line upward until both  $P$  and  $P'$  lie entirely below a line parallel to  $y = \frac{(n+2)}{(n+1)}x$ . For example, in Figure 4.7, we pushed the original line upward until  $P$  and  $P'$  were entirely below the red line. The resultant  $A_{n+1}$  path is highlighted in blue. Since  $(n + 1)$  and  $(n + 2)$  are relatively prime, no kinks can lie on a line parallel to  $y = \frac{(n+2)}{(n+1)}x$  except for each pair of kinks  $a_i, a'_i$ . Also, since the endpoints of the path  $P$  behave like kinks (the endpoint of  $P$  and the startpoint of  $P'$  forms a kink), we see that there are  $k + 1$  kinks total. Therefore, there is exactly one pair of  $a_i, a'_i$  out of the  $k + 1$  total kink pairs such that  $P$  and  $P'$  lie below the line

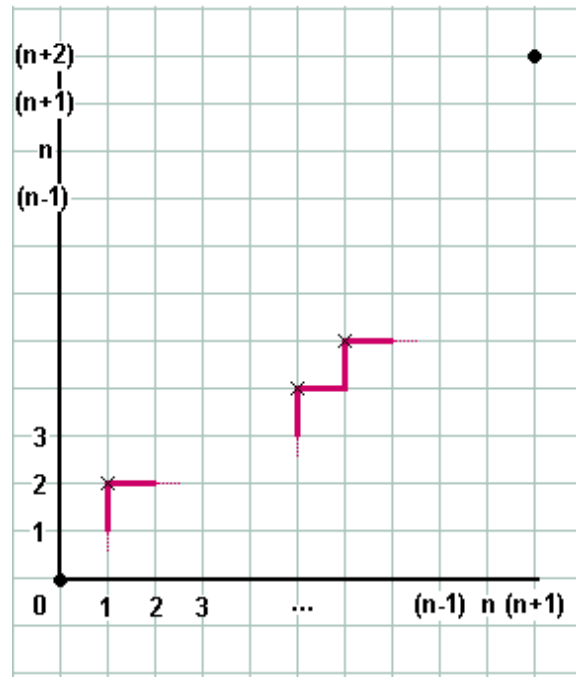


Figure 4.5: An acceptable arrangement of kinks.

connected by these endpoints.

In other words, we have constructed a path from  $(0, 0)$  and found that exactly one out of the  $k + 1$  pairs will construct a line which stays above the entire  $P - P'$  path. Therefore, exactly one out of every  $k + 1$  unrestricted paths will originally be an  $A_{n+1}$  path. Therefore, we must divide out by  $k + 1$ , yielding  $\binom{n}{k} = \frac{1}{k+1} \binom{n+1}{k} \binom{n}{k}$  total  $A_{n+1}$  paths with exactly  $k$  kinks. Since we established a simple bijection between  $A_{n+1}$  paths and Catalan paths, we have demonstrated that  $\binom{n}{k}$  counts the number of Catalan paths from  $(0, 0)$  to  $(n + 1, n + 1)$  with exactly  $k$  kinks.  $\square$

Immediately, we see that the triangular coefficient must be an integer, as it always counts a discrete number of Catalan paths. Also, we can now combinatorially prove another identity presented in the introduction.



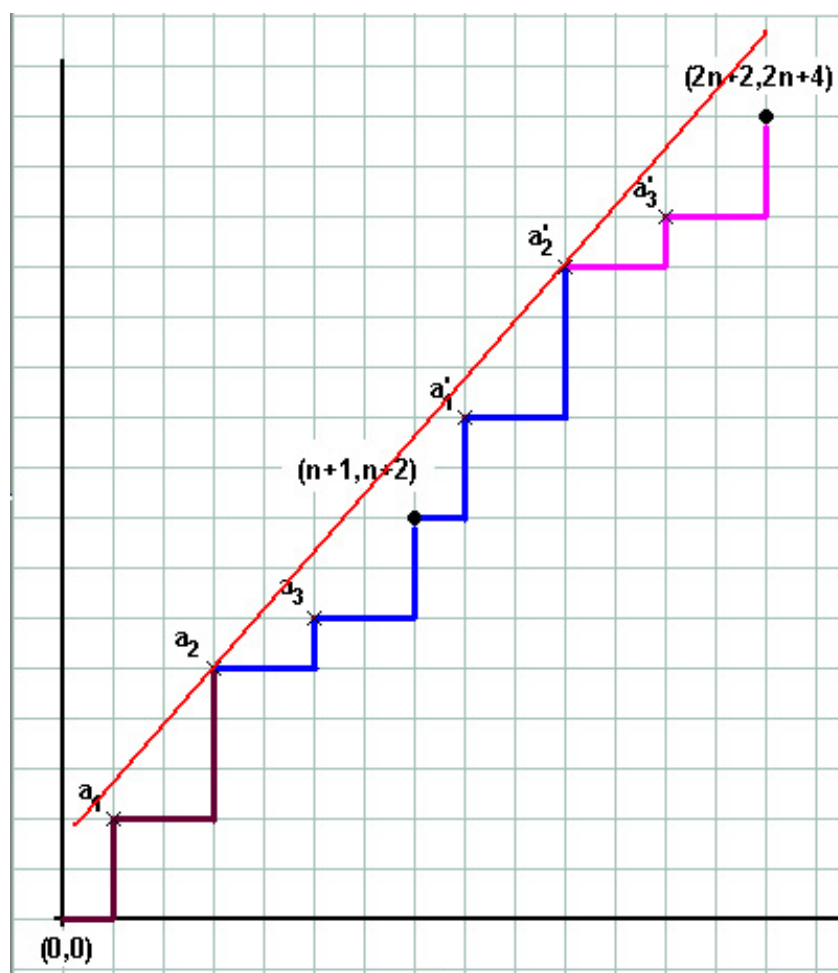


Figure 4.7: Exactly one kink pair will serve as endpoints for a line which stays completely above both  $P$  and  $P'$ .

### 4.3 Non-Intersecting Paths

The triangular coefficient can also be interpreted as counting a specific type of non-intersecting path. Before giving this interpretation, we introduce the notion of counting non-intersecting paths.

Consider simultaneous paths from starting points  $S_1$  and  $S_2$  to destinations  $D_1$  and  $D_2$ . Let these starting points and destinations be arranged as in Figure 4.8.

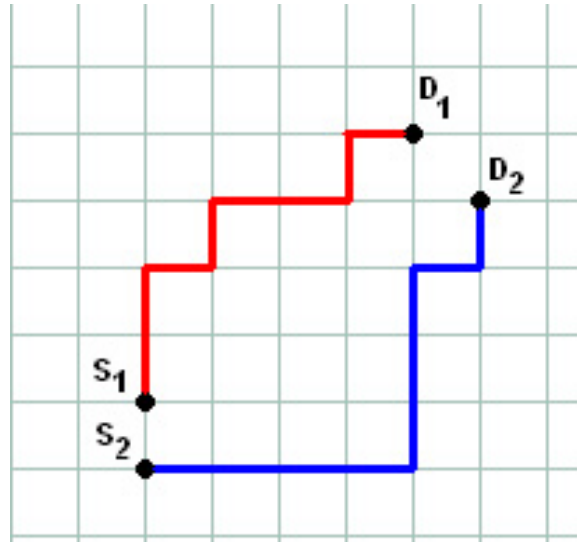


Figure 4.8: A simultaneous path from  $S_1$  to  $D_1$  and  $S_2$  to  $D_2$ .

Let the number of paths from starting points to destinations be given by the following table.

Total # of paths	$D_1$	$D_2$
$S_1$	A	B
$S_2$	C	D

Therefore, if we consider simultaneous paths from our starting points to our destinations, there are two possibilities:  $S_1$  to  $D_1$  and  $S_2$  to  $D_2$ , which we shall call a *standard* simultaneous path, or  $S_1$  to  $D_2$  and  $S_2$  to  $D_1$ , which we shall call a *crossover* simultaneous path. Note that a crossover path will always have some point of intersection between the two paths. Therefore, the total number of simultaneous paths is  $AD + BC$ .

Now, consider *intersecting* standard simultaneous paths, such as in Figure 4.9.

We claim that there are  $BC$  such paths with at least one intersection. Consider a standard simultaneous path with a point of intersection between the two paths. Let  $x$  be the first point of intersection of these two paths. After the point  $x$ , we swap the tails of the two paths. That is, our  $S_1$  path was originally going to  $D_1$ , but after  $x$ , the  $S_1$  path follows the one started by  $S_2$  and will end in  $D_2$ . Likewise,  $S_2$  was originally going to  $D_2$ ,

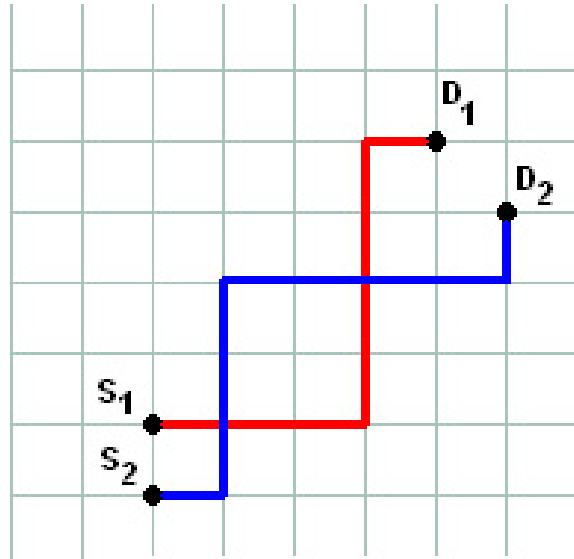


Figure 4.9: A standard path with an intersection.

but after  $x$ ,  $S_2$  will follow along  $S_1$ 's path to  $D_1$ . A visual example is provided in Figure 4.10.

In other words, since every crossover path had a point of intersection, we see that we have formed a bijection between standard paths with intersection and crossover paths. Therefore, there are  $BC$  standard paths with a point of intersection, yielding  $AD - BC$  standard paths without intersection. Note that we can encode this as the determinant of the following matrix.

$$\text{Nonintersecting standard paths} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}$$

We see that this matrix has the exact same format as the previous table. We use this notion of non-intersecting paths to prove the following theorem.

**Theorem 4.**

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{vmatrix} \binom{n}{k} & \binom{n}{k+1} \\ \binom{n+1}{k} & \binom{n+1}{k+1} \end{vmatrix}$$

*Proof.* We first consider the matrix. Using our non-intersecting path interpretation of the determinant, we simply need to establish which points are our starting points and destinations and the behavior of the paths. Let the

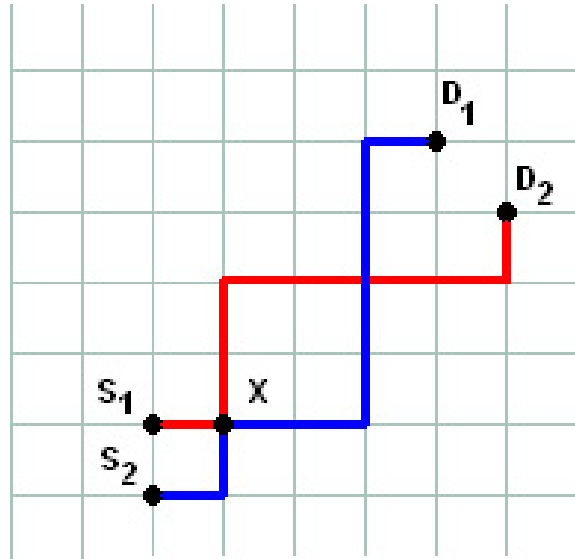


Figure 4.10: We swap the tails between that paths from Figure 4.9 to yield the above crossover path.

two starting points be  $(0,0)$  and  $(0,-1)$  and let the destinations be  $(k,n-k)$  and  $(k+1,n-k-1)$ . Also, we are counting the number of lattice paths that only move up and right. So, from  $(0,0)$  to  $(k,n-k)$ , we must perform  $k+(n-k)=n$  total moves, so if we choose which  $k$  are the right moves, there are  $\binom{n}{k}$  total such paths. Performing similar counting arguments for the other start/destination pairs yields the above matrix.

For example, if  $n=4$  and  $k=2$ , we could have such restricted paths as in Figure 4.11.

However note that in each case, if we draw a line segment from  $(0,0)$  to  $(0,-1)$  and close the top right corner by  $(k,n-k)$  and  $(k+1,n-k-1)$ , we form polyominoes of size  $n+2$ , as in Figure 4.12.

Therefore, the original matrix counts the number of polyominoes of size  $n+2$  with width  $k+1$ , where the *width* of a polyomino is the number of columns.

We now demonstrate a simple bijection between polyominoes of size  $n+2$  and Catalan paths to  $(n+1,n+1)$ . Consider a polyomino of size  $n+2$ . First, we break up the polyomino into its corresponding columns. Each column has two parameters: the height and the overlap from the previous column. For example, consider the third polyomino in Figure 4.13.

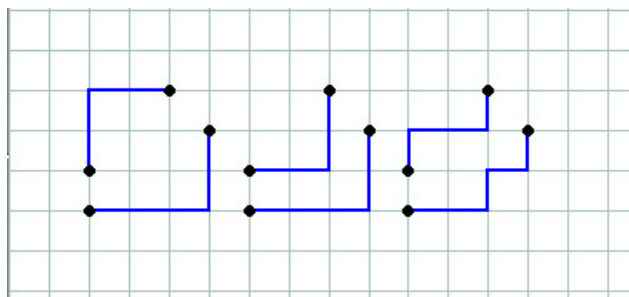


Figure 4.11: An example of restricted paths.

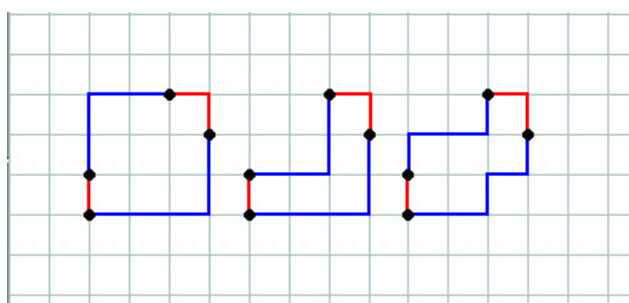


Figure 4.12: By closing the corners, we form polyominoes.

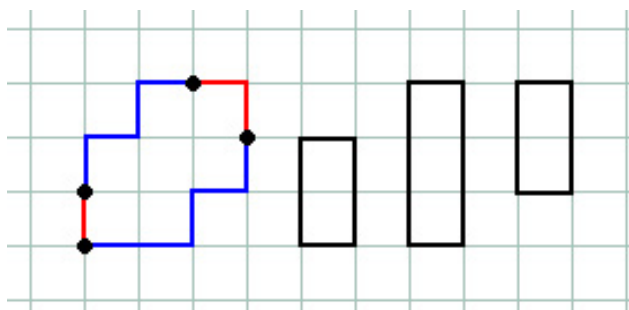


Figure 4.13: We can break up a polyomino into its individual columns.

Since every column must have an overlap of at least one (except for the first column), we say that the overlap parameter is one less than the actual overlap. So, in this example, the first column has a height 2 and no overlap parameter (as it was the first column), the second column has a height 3

and overlap 1, and the third column has a height 2 and overlap 1. Starting with the first column, we draw a right triangle with height and base equal to the height for each of the columns. Continuing with the above example, we would have the right triangles as seen in Figure 4.14.

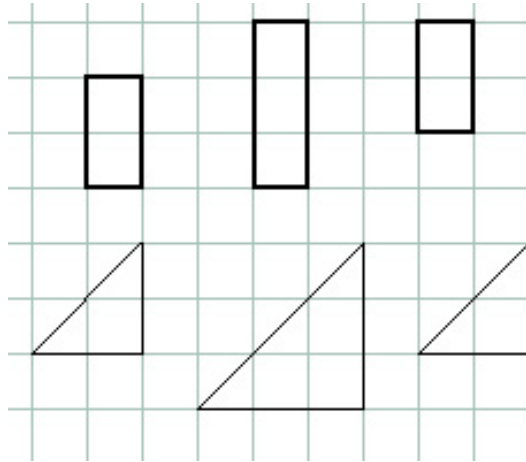


Figure 4.14: Each column of the polyomino corresponds to a triangle which will be used to form a Catalan path.

We will now place these triangles on the integer lattice to form a Catalan path. We place the first triangle (corresponding to the first column) flush at  $(0,0)$  and the  $y = x$  line. To place the next triangle, we look at the  $y$ -coordinate of our path endpoint and  $p$ , the value of the overlap parameter. We place the next triangle's base at  $(y - p)$  and flush with the  $y = x$  line.

Tracing along the envelope of these triangles yields a Catalan path from  $(0,0)$  to  $(n + 1, n + 1)$ . Note that the overlap will be at least zero, and thus we can never cross the  $y = x$  line. We see the triangles used to form a Catalan path in this fashion in Figure 4.15.

The number of kinks in each Catalan path is equal to one less than the number of columns of the corresponding polyomino. Therefore, by the above bijection, since the matrix on the right counts the number of polyominoes with  $k + 1$  columns, it also counts the number of Catalan paths with  $k$  kinks. Therefore

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{vmatrix} \binom{n}{k} & \binom{n}{k+1} \\ \binom{n+1}{k} & \binom{n+1}{k+1} \end{vmatrix}$$

as desired. □



$\begin{bmatrix} n \\ k \end{bmatrix}$ . The Schroder numbers are similar to Narayana numbers. They are restricted paths to  $(n, n, \dots, n)$  that lie in the same wedge as  $C(d, n)$  described in Chapter 3. However, Schroder numbers allow *any* movement along the unit cube. That is, in Narayana numbers, a path could move in any of  $X_1 = (1, 0, \dots, 0), X_2 = (0, 1, \dots, 0), \dots, X_d = (0, 0, \dots, 1)$ . In Schroder numbers, a path can move in any step of the form  $(\xi_1, \xi_2, \dots, \xi_d)$  where  $\xi_i \in \{0, 1\}$ .



## Chapter 5

# Baxter Coefficients

### 5.1 Introduction

In the previous chapter, we considered the triangular coefficient. Since  $t_n = \binom{n+1}{2}$  we can rewrite the triangular coefficient as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{t_n t_{n-1} \cdots t_{n-k+1}}{t_k \cdots t_1} = \frac{\binom{n+1}{2} \binom{n}{2} \cdots \binom{n-k+2}{2}}{\binom{k+1}{2} \cdots \binom{2}{2}}.$$

We find surprising results when we increment each term in each binomial coefficient, replacing  $\binom{i}{2}$  by  $\binom{i+1}{2+1} = \binom{i+1}{3}$ . We define

$$\begin{bmatrix} n \\ k \end{bmatrix}_3 = \frac{\binom{n+2}{3} \binom{n+1}{3} \cdots \binom{n-k+3}{3}}{\binom{k+2}{3} \cdots \binom{3}{3}}.$$

This object generates its own triangle

$$\begin{array}{cccccc} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 4 & 1 & & & \\ 1 & 10 & 10 & 1 & & \\ 1 & 20 & 50 & 20 & 1 & \\ 1 & 35 & 175 & 175 & 35 & 1 \end{array}$$

We again are surprised to see that this new coefficient is always an integer. More importantly, though, when we sum across the rows, we have the sequence 1, 2, 6, 22, 92, 422 . . . or that

$$\sum_{k \geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_3 = B_n.$$

Therefore, we call  $\begin{bmatrix} n \\ k \end{bmatrix}_3$  a Baxter coefficient.

## 5.2 Interesting Correspondences

We begin by proving the fundamental observation of the inner triangle.

**Theorem 6.**

$$\sum_{k \geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_3 = B_n.$$

*Proof.*

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}_3 &= \frac{\binom{n+2}{3} \binom{n+1}{3} \cdots \binom{n-k+3}{3}}{\binom{k+2}{3} \cdots \binom{3}{3}} \\ &= \frac{[(n+2)(n+1)(n)][(n+1)(n)(n-1)] \cdots [(n-k+3)(n-k+2)(n-k+1)]}{[(k+2)(k+1)(k)] \cdots [4 \cdot 3 \cdot 2][3 \cdot 2 \cdot 1]} \\ &= \frac{2}{(k+1)^2(k+2)} \frac{(n+2) \cdots (n-k+3)}{(k) \cdots (1)} \frac{(n+1) \cdots (n-k+2)}{(k) \cdots (1)} \frac{(n) \cdots (n-k+1)}{(k)(k-1) \cdots (1)} \\ &= \frac{2}{(k+1)^2(k+2)} \binom{n+2}{k} \binom{n+1}{k} \binom{n}{k} \\ &= \frac{\binom{n+2}{k} \binom{n+1}{k} \binom{n}{k}}{\binom{k+1}{1} \binom{k+2}{2}} \\ &= \frac{\binom{n+1}{k} \binom{n+1}{k+1} \binom{n+1}{k+2}}{\binom{n+1}{1} \binom{n+1}{2}} \end{aligned}$$

Therefore, as noted in Chapter 3, summing over a row yields

$$\sum_{k=1}^n \frac{\binom{n+1}{k-1} \binom{n+1}{k} \binom{n+1}{k+1}}{\binom{n+1}{1} \binom{n+1}{2}} = B_n.$$

as desired.  $\square$

Similar to the triangle coefficient, we see that the Baxter coefficient can also be found by taking a determinant in Pascal's triangle.

**Theorem 7.**

$$\left[ \begin{matrix} n \\ k \end{matrix} \right]_3 = \begin{vmatrix} \binom{n}{k} & \binom{n}{k+1} & \binom{n}{k+1} \\ \binom{n+1}{k} & \binom{n+1}{k+1} & \binom{n+1}{k+2} \\ \binom{n+2}{k} & \binom{n+2}{k+1} & \binom{n+2}{k+2} \end{vmatrix}$$

*Proof.* Although we can perform algebraic manipulations to prove this fact, this fact is an immediate corollary of Dulucq and Guibert’s work [2]. In their paper *Baxter Permutations*, they demonstrate that the number of Baxter permutations with  $k$  rises is

$$\begin{vmatrix} \binom{n}{k} & \binom{n}{k+1} & \binom{n}{k+1} \\ \binom{n}{k} & \binom{n}{k+1} & \binom{n}{k+2} \\ \binom{n}{k} & \binom{n}{k+1} & \binom{n}{k+2} \end{vmatrix}$$

In other words, as discussed in Chapter 4, this matrix counts the number of non-intersecting paths with starting points  $S_1 = (0, 0), S_2 = (1, -1), S_3 = (2, -2)$  and destination points  $D_1 = (k, n - k), D_2 = (k + 1, n - k - 1), D_3 = (k + 2, n - k - 2)$ . Figure 5.1 gives a visual example of three non-intersecting paths with these start and end points.

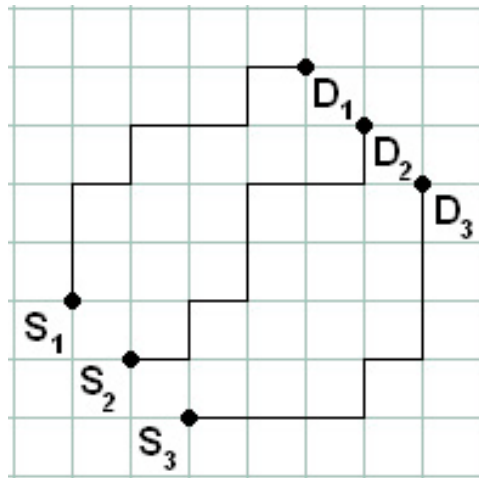


Figure 5.1: Three non-intersecting paths.

However, note that, since we may only make right and up moves, the number of non intersecting paths does not change if we move  $S_3$  to  $(0, -2)$  and  $S_2$  to  $(0, -1)$  as in Figure 5.2.



Although no connection has been found, we are interested to see the relationship between three-dimensional Catalan paths and Baxter coefficients, as both formulae look extraordinarily similar:

$$C(3, n) = \frac{\binom{3n}{n, n, n}}{\binom{n+1}{1} \binom{n+2}{2}}$$

$$\left[ \begin{matrix} n \\ k \end{matrix} \right]_3 = \frac{\binom{n}{k} \binom{n+1}{k} \binom{n+2}{k}}{\binom{k+1}{1} \binom{k+2}{2}}.$$



## Chapter 6

# The Fibonomial Coefficient

Replacing each factor of the binomial coefficient with a Fibonacci number yields the Fibonomial coefficient. We define  $\begin{bmatrix} n \\ k \end{bmatrix}_F$  to be

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_k \cdots F_1}.$$

where  $F_0 = 0$ ,  $F_1 = 1$  and  $F_{n+1} = F_n + F_{n-1}$  for  $n \geq 2$ . Again, the Fibonomials have their own triangle,

$$\begin{array}{cccccc} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 1 & 1 & & & \\ 1 & 2 & 2 & 1 & & \\ 1 & 3 & 6 & 3 & 1 & \\ 1 & 5 & 15 & 15 & 5 & 1 \end{array}.$$

We are surprised to see that the Fibonomial coefficient is always an integer.

Algebraically and in the triangle we also see that  $\begin{bmatrix} n \\ k \end{bmatrix}_F = \begin{bmatrix} n \\ n-k \end{bmatrix}_F$ .

To prove these facts, we use the well known fact that that  $f_n = F_{n+1}$ , the shifted Fibonacci numbers, count the number of ways to tile a board of length  $n$  with squares and dominoes.

### 6.1 Always an Integer

**Theorem 8.**  $\begin{bmatrix} n \\ k \end{bmatrix}_F$  is an integer.

*Proof.* Using the shifted Fibonacci numbers, we have  $\begin{bmatrix} n \\ k \end{bmatrix}_F = \frac{f_{n-1}f_{n-2}\dots f_{n-k}}{f_{k-1}\dots f_0}$ .

To prove that  $\begin{bmatrix} n \\ k \end{bmatrix}_F$  is always an integer, we demonstrate that there are always tilings of length  $k - 1, k - 2, \dots, 1, 0$  in tilings of length  $n - 1, n - 2, \dots, n - k$ . We call the tilings from the denominator the *sub-tilings* and the tilings from the numerator the *pyramid* tilings. Just as we would cancel equal terms algebraically, we remove any pyramid tilings and sub-tilings of equal length as a trivial initial search. For the remainder of the tilings we, not surprisingly, arrange the pyramid tilings in a pyramid as in Figure 6.1.

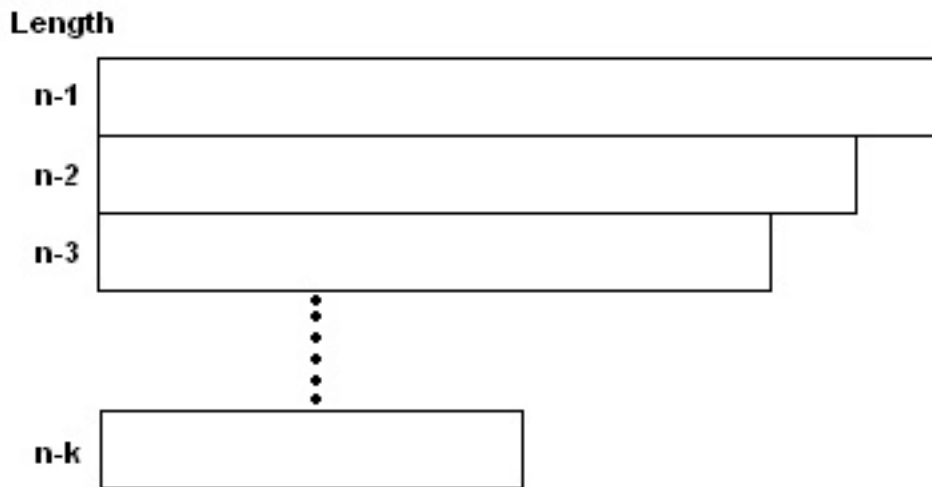


Figure 6.1: A pyramid arrangement of the pyramid tilings.

We will find exactly one sub-tiling in each pyramid tiling. We begin by examining the tilings one at the time starting at the top of the pyramid. First, we determine whether the  $(n - 1)$ -tiling is breakable at cell  $k - 1$ . If the  $(n - 1)$ -tiling is breakable at  $k - 1$ , we have succeeded in finding our first sub-tiling. So, we remove the  $(n - 1)$ -tiling from the pyramid and determine whether the next tiling is breakable at  $k - 2$ , as demonstrated in Figure 6.2.

In other words, if we have found  $i$  sub-tilings, we can continue look-

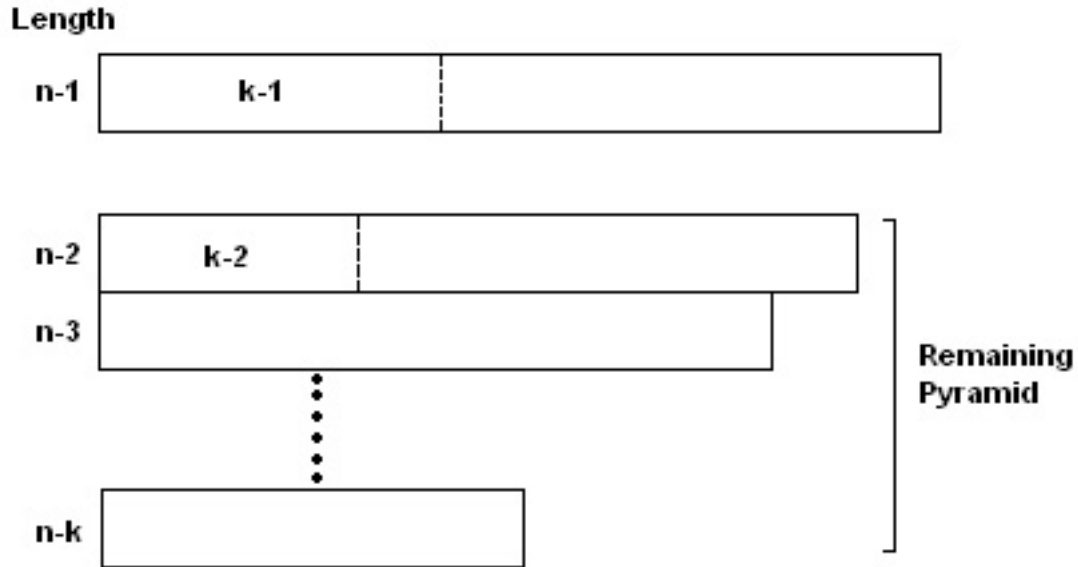


Figure 6.2: If the topmost tiling is breakable at  $k - 1$ , we examine the rest of the pyramid for the remaining sub-tilings.

ing for whether the topmost pyramid tiling is breakable at  $k - (i + 1)$ . However, we encounter a problem when the topmost pyramid tiling is *not* breakable at  $k - (i + 1)$ , in other words a domino is centered over cell  $k - (i + 1)$ . In this case, we take the portion of the topmost tiling which lies to the right of the domino and move it to the bottom of the pyramid, as in Figure 6.3.

Again, if we have found  $i$  sub-tilings, there are still  $k - i$  tilings remaining in the pyramid and  $k - i$  sub-tilings left to find. By performing the move in Figure 6.3, we still have a pyramid with  $k - i$  tilings, but the pyramid is slightly smaller than before, as in Figure 6.4.

At each step in the process, if we have found  $i$  sub-tilings, we determine whether the topmost pyramid tiling is breakable at  $k - (i + 1)$ . We either have a success and our pyramid shrinks by one tiling, or we have a domino and move the remaining topmost tiling to the bottom of the pyramid. In either case, the topmost tiling length will decrease by exactly one after each iteration. In the worst case scenario, if we have found  $i$  sub-tilings, we are left with a pyramid whose tilings are length  $k - (i + 1), k - (i + 2), \dots, 1, 0$ , giving us all the remaining sub-tilings. Therefore, we can

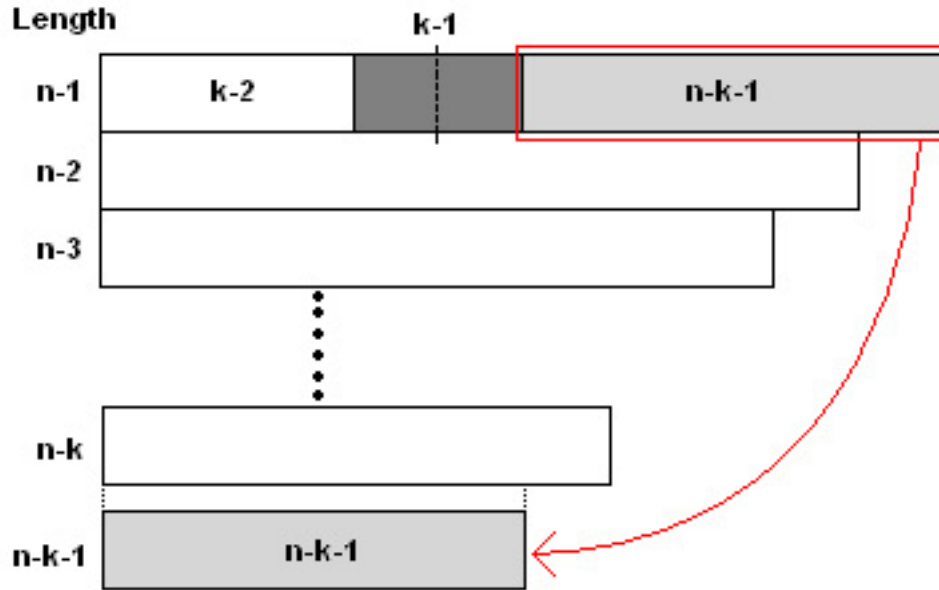


Figure 6.3: If the topmost tiling isn't breakable at  $k - 1$ , we move the remainder of the tiling to the bottom of the pyramid.

always find tilings of lengths  $0, 1, \dots, k - 1$  in any set of tilings of lengths  $n - k, n - k + 1, \dots, n - 2, n - 1$ , so we know that  $\begin{bmatrix} n \\ k \end{bmatrix}_F$  must always be an integer, as desired. □

## 6.2 Symmetry

**Theorem 9.**

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \begin{bmatrix} n \\ n - k \end{bmatrix}_F$$

*Proof.* Suppose without loss of generality that  $k \leq \frac{n}{2}$ . Consider  $\begin{bmatrix} n \\ k \end{bmatrix}_F$ . We know that when we perform the initial search, we do not trivially find any tilings. That is, since  $k \leq \frac{n}{2}$ , there will be no sub-tilings and pyramid tilings

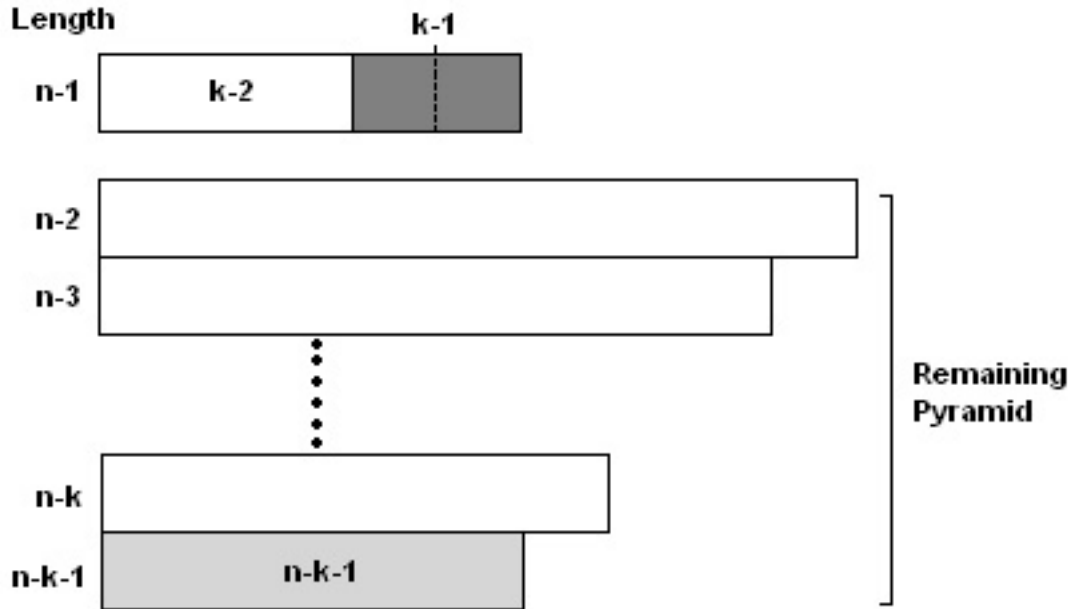


Figure 6.4: The new pyramid has the same number of tilings as the previous iteration, but the overall size is decreased.

of equal length. However, when we consider  $\begin{bmatrix} n \\ n-k \end{bmatrix}_F$ , we find exactly  $n - 2k$  sub-tilings and pyramid tilings of equal length in our initial search. The resultant initial pyramid is the exact same pyramid as constructed when we considered  $\begin{bmatrix} n \\ k \end{bmatrix}_F$ . Consequently, we see an obvious bijection between the pyramids of  $\begin{bmatrix} n \\ k \end{bmatrix}_F$  and those of  $\begin{bmatrix} n \\ n-k \end{bmatrix}_F$ . Therefore, we know that  $\begin{bmatrix} n \\ k \end{bmatrix}_F = \begin{bmatrix} n \\ n-k \end{bmatrix}_F$ .

### 6.3 Combinatorially Proving other Results

In [5], Jaroslav Seibert proved three identities of the Fibonomial coefficient using generating functions, analysis, and algebraic manipulation. We provide an alternative proof of one of his results below.

**Theorem 10.** For  $n$  odd,

$$\sum_{i=0}^m (-1)^{\frac{i}{2}(n+i)} \begin{bmatrix} n \\ i \end{bmatrix}_F = 0$$

We have already seen that  $\begin{bmatrix} n \\ k \end{bmatrix}_F = \begin{bmatrix} n \\ n-k \end{bmatrix}_F$ . Therefore, we must simply demonstrate that  $\frac{i}{2}(n+i)$  and  $\frac{n-i}{2}(n+(n-i)) = \frac{n-i}{2}(2n-i)$  have opposite parity. To do this, we demonstrate that the difference between  $\frac{1}{2}(i)(n+i)$  and  $\frac{1}{2}(n-i)(2n-i)$  is always odd, implying that  $(i)(n+i)$  and  $(n-i)(2n-i)$  have opposite parity.

$$\begin{aligned} \frac{1}{2}(n-i)(2n-i) - \frac{1}{2}(i)(n+i) &= \frac{1}{2}(2n^2 - ni - 2ni + i^2 - ni - i^2) \\ &= \frac{1}{2}(2n^2 - 4ni) = n^2 - 2ni \end{aligned}$$

Since  $n^2$  is always odd, we see that  $n^2 - 2ni$  is always odd, and therefore  $\frac{i}{2}(n+i)$  and  $\frac{n-i}{2}(n+(n-i)) = \frac{n-i}{2}(2n-i)$  have opposite parity.

We demonstrate that exactly one of  $(i)(n+i)$  and  $(n-i)(2n-i)$  will be congruent to 0 mod 4 and the other will be congruent to 2 mod 4. We immediately note that, because  $n$  is odd,  $i$  and  $n+i$  have opposite parity, as do  $(n-i)$  and  $(2n-i)$ . Therefore  $\begin{bmatrix} n \\ k \end{bmatrix}_F$  and  $\begin{bmatrix} n \\ n-k \end{bmatrix}_F$  will be weighted oppositely. So, summing across an odd row will always yield 0, as desired.

Note that this identity does not hold for even rows of  $n$ . For example, consider the fourth row of the Fibonomial Triangle: 1 3 6 3 1. Not only does the identity not hold for this row, but it is impossible to assign any combination of positive or negative weights to the terms to yield a sum of zero.  $\square$

## 6.4 Open Identities

In [5], Jaroslav Seibert provided two other identities for which we have yet to determine combinatorial proofs. The identities, where  $L_n$  is the  $n^{\text{th}}$  Lucas number, are

$$\begin{aligned} \sum_{i=0}^m (-1)^{\frac{i}{2}(2l+i+1)} \frac{F_{(k-i)(k-2l)}}{F_{(k-2l)}} \begin{bmatrix} k+1 \\ i \end{bmatrix} &= 0 \\ \sum_{i=0}^m (-1)^{\frac{i}{2}(2l+i+(-1)^k)} L_{(k-2l)(i+n)} \begin{bmatrix} k+1 \\ i \end{bmatrix} &= 0. \end{aligned}$$

## Chapter 7

# Future Directions

Though many of the identities for the Baxter and triangular numbers already exist through research into Schroeder and Narayana numbers, the previous research relied primarily on mechanical and somewhat confusing generating function arguments. Fortunately, the combinatorial proofs in chapters 4 and 5 provide solid intuitive explanations for the identities and relationships.

For future work, we are curious to see whether continuing the theme from the triangular and Baxter coefficients holds true, specifically

$$\begin{bmatrix} n \\ k \end{bmatrix}_a = \frac{\binom{n+a-1}{a} \binom{n+a-2}{a} \cdots \binom{n-k+a}{a}}{\binom{k+a-1}{a} \cdots \binom{a}{a}} = \begin{vmatrix} \binom{n}{k} & \cdots & \binom{n}{k+a-1} \\ \vdots & \ddots & \vdots \\ \binom{n+a-1}{a} & \cdots & \binom{n+a-1}{k+a-1} \end{vmatrix}.$$

Since taking  $2 \times 2$  determinants and  $3 \times 3$  determinants yield partitions of Catalan and Baxter numbers, respectively, we are interested to explore whether the  $n \times n$  case provides anything more than  $n$  non-intersecting paths.

We also are curious to explore a simple combinatorial proof for a  $d$ -dimensional Catalan path. More importantly, we wish to explore the relationship between a  $d \times d$  determinant in Pascal's triangle and a  $d$ -dimensional Catalan path. We already illustrated many connections for two dimensions:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{\binom{n}{k} \binom{n+1}{k}}{\binom{k+1}{1}} \quad \text{and} \quad C(2, n) = \frac{\binom{2n}{n}}{\binom{n+1}{1}}.$$

Here, we see that the triangular coefficient partitions Catalan paths by the number of kinks. However, we are unable to fully explain the similar form

for three dimensional case:

$$\begin{bmatrix} n \\ k \end{bmatrix}_3 = \frac{\binom{n}{k} \binom{n+1}{k} \binom{n+2}{k}}{\binom{k+1}{1} \binom{k+2}{2}} \quad \text{and} \quad C(3, n) = \frac{\binom{3n}{n, n, n}}{\binom{n+1}{1} \binom{n+2}{2}}.$$

Last, we wish to combinatorially prove the final two identities provided by Seibert, and perhaps find a way to generate the Fibonomial coefficient using Pascal's triangle, as we did for the the triangular and Baxter coefficient.

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