Abelian Sandpile Model: Symmetric Sandpiles

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Self Organized Criticality

In an equilibrium system the critical point is reached by tuning a control parameter precisely. Example: Melting water.

Definition

Self-Organized Criticality *A phenomenon of a non-equilibrium system with a critical point as attractor. Behavior of critical points same as phase transition, but without control parameters.*

Self Organized Criticality

Sand piles exhibit this behavior: static periods with intermittent sand slides. Catastrophes are inevitable, the product of minor events in the past.

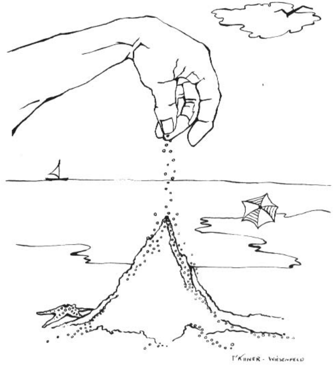


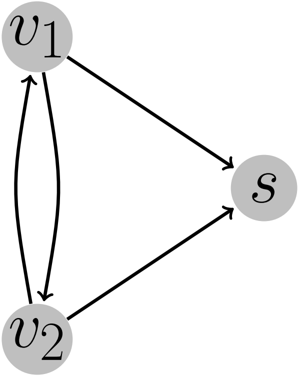
Figure: From *How Nature Works*, Per Bak.

Self Organized Criticality

- Punctuated Equilibrium (Sand avalanches correspond to cladogenesis, rapid events of branching speciation)
- Modeling Evolution (No need for meteors to explain dinosaur extinction)
- The Brain (Thought is a critical state, an avalanche triggered by visual stimulus or another thought)
- Economics (Network of consumers and producers, supply and demand avalanches. Critical economy vs. equilibrium economy)

The Model

Directed graph, $\Gamma = (V, E, s)$ (multiple edges and self loops OK).



- d_{v_1} = outdegree$(v_1) = 2$
- $d_s = 0$
- s is a *global sink*

The Model

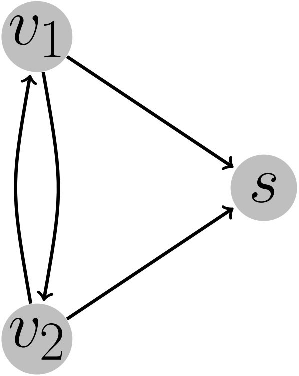
Directed graph, $\Gamma = (V, E, s)$ (multiple edges and self loops OK).

- Laplacian

$$\Delta = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

- Reduced Laplacian

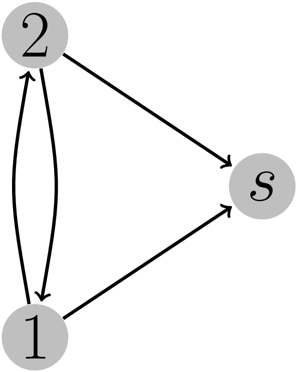
$$\Delta' = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$



Sandpiles

Sandpile: Integer weighting on the non-sink vertices of Γ

- $\sigma \in \mathbb{N}^2$
- $\sigma = (2, 1)$
- $\sigma(v_1) = 2$, $\sigma(v_2) = 1$
- σ *unstable* at v if $\sigma(v) \geq d_v$
- σ is *unstable* at v_1

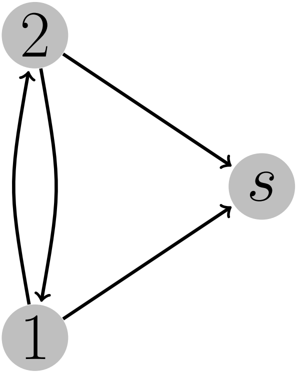


Stabilization

Firing unstable vertices: Recall the Reduced Laplacian

$$\Delta' = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$\sigma = \quad (2, 1)$



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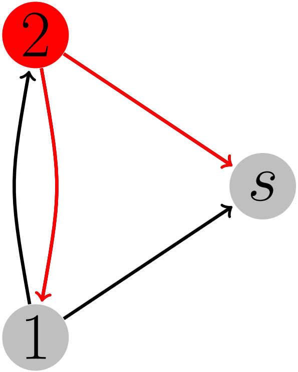
$$\begin{array}{rr} \sigma = & (2,1) \\ - & (2,-1) \\ \hline & (0,2) \end{array}$$



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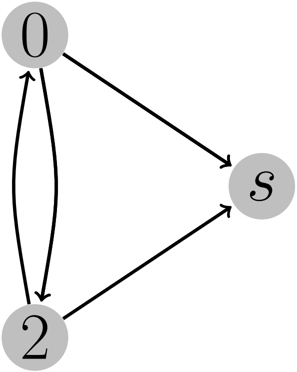

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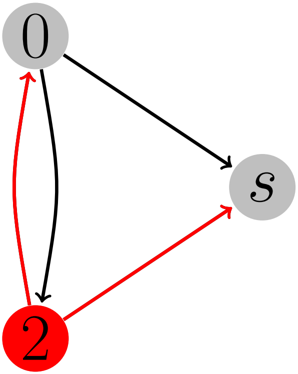

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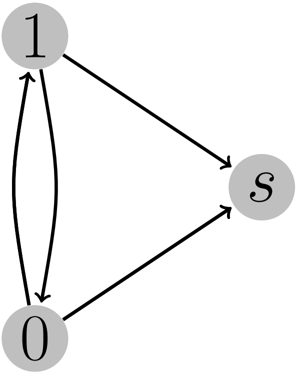
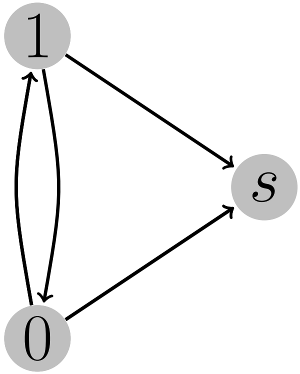
σ° is the *stabilization* of σ.



Stabilization

Theorem

If Γ has a global sink, every configuration on Γ stabilizes.



Sand Addition Operator

Definition

For $v \in V \backslash s$, the **sand addition operator** E_v *acts on configurations as follows:*

$$E_v \sigma = (\sigma + 1_v)^\circ$$

For digraph with global sink, sand addition operator commutes, i.e.

$$E_w E_v \sigma = (\sigma + 1_w + 1_v)^\circ = (\sigma + 1_v + 1_w)^\circ = E_v E_w \sigma$$

Applying a sequence of sand additions to σ yields same result as adding all sand simultaneously and then stabilizing.

The Sandpile Group

- Let Δ' denote the reduced Laplacian of Γ, with n the number of non-sink vertices. The Sandpile group of Γ is given by:

$$S(\Gamma) = \mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}\Delta'(\Gamma)$$

- $S(\Gamma)$ is the integer row span of the reduced Laplacian.
- $|S(\Gamma)| = \det(\Delta')$

Recurrent Configurations

Definition

A configuration σ is **accessible** *if for all configurations α, there exists a β such that $\alpha + \beta \to \sigma$. If σ is stable and accessible, then σ is* **recurrent***.*

- Every equivalence class of $S(\Gamma)$ contains a unique recurrent configuration.
- Set of all recurrent configurations on Γ forms an abelian group under $(\sigma, \sigma') \mapsto (\sigma + \sigma')^\circ$ and is isomorphic to $S(\Gamma)$.

The Identity Sandpile

$$I = (\sigma - \sigma^\circ)^\circ$$

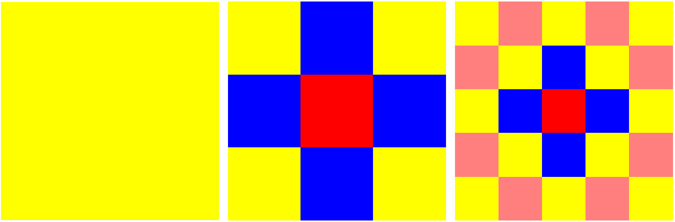
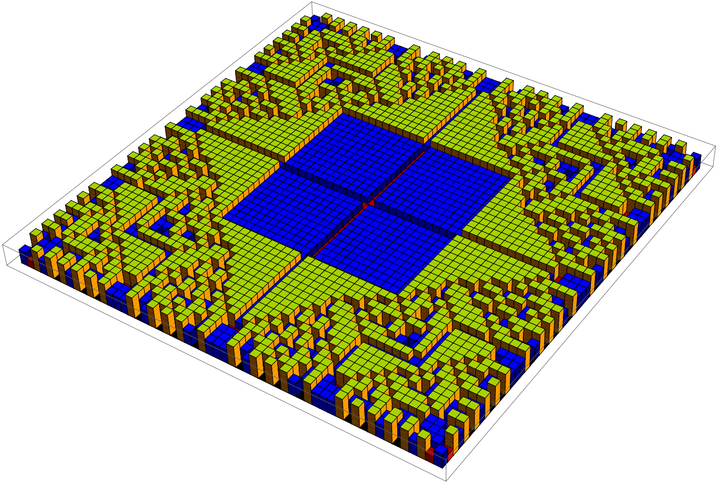


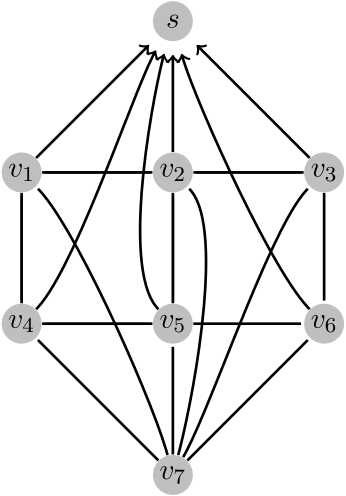
Figure: Identities for the 2×2 grid, 3×3 grid, and 5×5. Colors: red = 1, blue = 2, yellow = 3, pink =4.

The Identity Sandpile: 57 by 57 Grid



The Six Grid

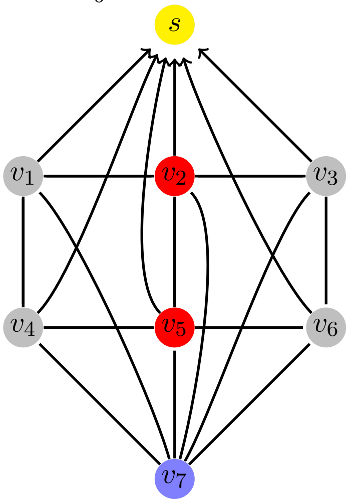
$\Gamma_6 =$



$$\tilde{\Delta} = \begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & -1 \\ -1 & 5 & -1 & 0 & -1 & 0 & -1 \\ 0 & -1 & 4 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 5 & -1 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 6 \end{pmatrix}$$

Group Action Partitions Vertices into Equivalence Classes

G on $\Gamma_6 =$



Sum the rows in each equivalence classes:

$$\tilde{\Delta} = \begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & -1 \\ -1 & 5 & -1 & 0 & -1 & 0 & -1 \\ 0 & -1 & 4 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 5 & -1 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 6 \end{pmatrix}$$

Obtaining the Quotient Graph

Equivalence Class	$(v_1, v_2, v_3, v_4, v_5, v_6, v_7)$
$\{v_1, v_3, v_4, v_6\}$	$(3, -2, 3, 3, -2, 3, -4)$
$\{v_2, v_5\}$	$(-1, 4, -1, -1, 4, -1, -2)$
$\{v_7\}$	$(-1, -1, -1, -1, -1, -1, 6)$

$$\Delta_q = \begin{pmatrix} 3 & -2 & -4 \\ -1 & 4 & -2 \\ -1 & -1 & 6 \end{pmatrix} \qquad \Delta_q^T = \begin{pmatrix} 3 & -1 & -1 \\ -2 & 4 & -1 \\ -4 & -2 & 6 \end{pmatrix}$$

Let Γ_q denote the graph described by Δ_q^T.

Symmetric Elements

Definition

$\sigma \in S(\Gamma_6)$ *is symmetric if it is of the form:*

$$(\sigma(v_1), \sigma(v_2), \sigma(v_1), \sigma(v_1), \sigma(v_2), \sigma(v_1), \sigma(v_7))$$

- The symmetric elements of $S(\Gamma_6)$ form a subgroup, $S(\Gamma_6)^G$.
- We also calculate the recurrent configurations that comprise $S(\Gamma_q)$ for the quotient graph.

Symmetric Subgroup and Laplacian of the Quotient Graph

- The sandpile group $S(\Gamma_q)$ has the same number of elements as the symmetric subgroup of the original graph!
- $|S(\Gamma_q)|$ is given by the determinant of Δ_q^T.

Problems with the Quotient Graph

- The elements of $S(\Gamma_q)$ do not match the elements of $S(\Gamma_6)^G$.
- Why did we need to use the transpose?
- Rows of Δ_q do not sum to zero.
- Δ_q should treat the entire equivalence class as a vertex.

Rescaling the Laplacian

$$\Delta_q = \begin{pmatrix} 3 & -2 & -4 \\ -1 & 4 & -2 \\ -1 & -1 & 6 \end{pmatrix} \xrightarrow{reweighted} \quad \Delta_q' = \begin{pmatrix} 12 & -4 & -4 \\ -4 & 10 & -2 \\ -4 & -2 & 6 \end{pmatrix}$$

- Note that Δ_q' is symmetric like a Laplacian should be.
- $|S(\Gamma_q')| = 352$ as opposed to 30. Too many elements!
- Configurations can have up to 11 sand grains on a vertex and be stable.

...And Normalizing?

$$\Delta'_q = \begin{pmatrix} 12 & -4 & -4 \\ -4 & 10 & -2 \\ -4 & -2 & 6 \end{pmatrix} \quad \stackrel{/4}{\longrightarrow} \quad \|\Delta'_q\|_{/4} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 5/2 & -1/2 \\ -1 & -1/2 & 3/2 \end{pmatrix}$$

$$\stackrel{/2}{\longrightarrow} \quad \|\Delta'_q\|_{/2} = \begin{pmatrix} 6 & -2 & -2 \\ -2 & 5 & -1 \\ -2 & -1 & 3 \end{pmatrix}$$

- The graph corresponding to $\|\Delta'_q\|_{/4}$ has only 11 elements.
- The graph corresponding to $\|\Delta'_q\|_{/2}$ has 44 elements.
- Straying too far from our original construction.

Future Exploration

- Does there exist an isomorphism between $S(\Gamma)^G$ and $S(\Gamma_q)$?
- More generalized notion of firing that allows us to dip negative

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Acknowledgements Questions

Any Questions?

- Professor Su (Advising)
- Professor Perkinson (Advising)