

RESEARCH PROPOSAL: THE SYMMETRIC SANDPILE SUBGROUP OF THE ABELIAN SANDPILE MODEL

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ABSTRACT. The Abelian Sandpile Model, or chip-firing game, is a cellular automaton model on finite directed graphs often used to describe the phenomenon of self-organized criticality. Here we define the notion of a symmetric sandpile configuration, and show that such configurations form a subgroup of the sandpile group. Using the insight gained, we explore various properties of this symmetric sandpile subgroup and attempt to describe the square that appears in the identity sandpile of the $n \times n$ grid with wired boundary.

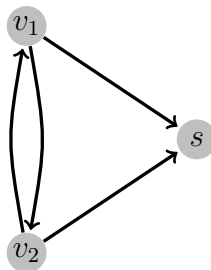
1. INTRODUCTION

The abelian sandpile model or chip-firing game has its roots in the study of self-organized criticality. A critical point in a physical system is a point where the system experiences radical behavioral or structural change, for example the phase transition of ice to water. Here temperature would be a control parameter that would allow a scientist to manipulate the system dynamics. A system that exhibits self-organized criticality, however, will achieve its critical state independent of any control parameter. In 1987, Bak, Tang and Wiesenfeld published a paper introducing a cellular automaton model that presents an example of a self-organized critical system: a sand pile [2]. Self-organized criticality has been used to describe systems like forest fires, earth-quakes and stock-market fluctuations [1]. Understanding such systems, then, would have extensive physical application.

The abelian sandpile model was first developed by Dhar [5]. Consider a finite, directed graph $\Gamma = (V, E, s)$ with multiple edges and self-loops allowed. Let d_v denote the out-degree of the vertex v . The graph contains a designated vertex s called the sink whose out degree is zero, $d_s = 0$. Let Δ denote the usual graph laplacian, and $\tilde{\Delta}$ denote the *reduced laplacian*, the matrix obtained by deleting the row and column corresponding to the sink in the laplacian. We compute the laplacian and reduced laplacian for a small example graph below:

$$\Delta = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{\Delta} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$



A sandpile configuration on a graph with n non-sink vertices is a vector of integer weights on the non-sink vertices, $\sigma \in \mathbb{N}^n$. A configuration σ is *unstable* at a vertex v if $\sigma(v) \geq d_v$. That is, if the number of grains of sand on a vertex equals or exceeds the out-degree of the vertex, then the vertex *topples* or *fires* adding one grain of sand each of its adjacent vertices. Firing an unstable vertex v is the same as subtracting the row of the reduced laplacian corresponding to v . Through a series of firings, every configuration *stabilizes* to σ° , and this stable configuration is independent of the firing order. We define addition of sandpiles to be $(\sigma_1 + \sigma_2)^\circ$ where $+$ denotes pairwise addition of the weights.

The maximum stable configuration σ_{\max} is given by $\sigma_{\max}(v) = d_v - 1$, i.e. every vertex has the most sand it can hold without toppling. *Recurrent* configurations are precisely those configurations that can be reached by adding sand to σ_{\max} and stabilizing. This collection of recurrent configurations forms a group under sandpile addition and is denoted, $S(\Gamma)$. The identity sandpile, $e \in S(\Gamma)$ can be computed as follows: $e = (\sigma - \sigma^\circ)^\circ$ for any $\sigma \in \mathbb{N}^n$. Considering $n \times n$ graphs, the identity sandpile emerges as an interesting object of study.

2. PROPOSED QUESTION

We say an $n \times n$ grid graph has *wired boundary* if it is undirected and each corner vertex has an edge of weight 2 going to the sink, and all other vertices on the boundary of the grid have an edge of weight 1 to the sink. Such boundary conditions give all non-sink vertices degree 4 [6]. The identity sandpile for grid graphs of various dimensions is depicted in Figures 1 and 2. In all of these pictures of identities, we note a level square in the center of the configuration. Also note that all of these identity configurations are highly symmetric. This brings us to the following problem:

Problem 1. *Prove the existence of the large square in the center of the $n \times n$ grid graph with wired boundary.*

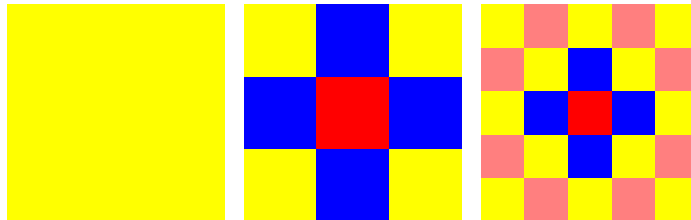


FIGURE 1. Identities for the 2×2 grid, 3×3 grid, and 5×5 . Colors: red = 1, blue = 2, yellow = 3. Generated in Mathematica.

This open question is proposed in [6], and was first brought to my attention by David Perkinson (Reed College). We next develop an approach to solving this problem.

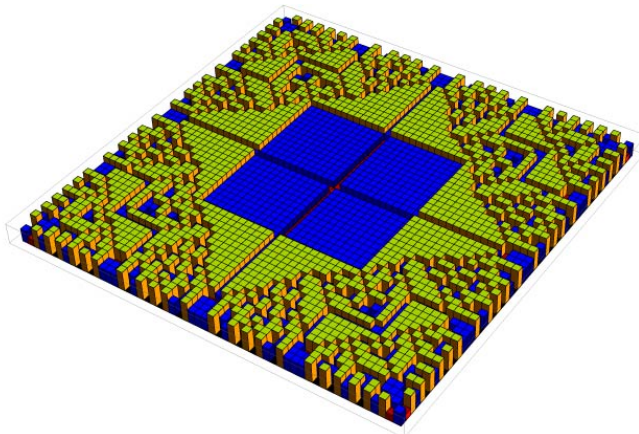
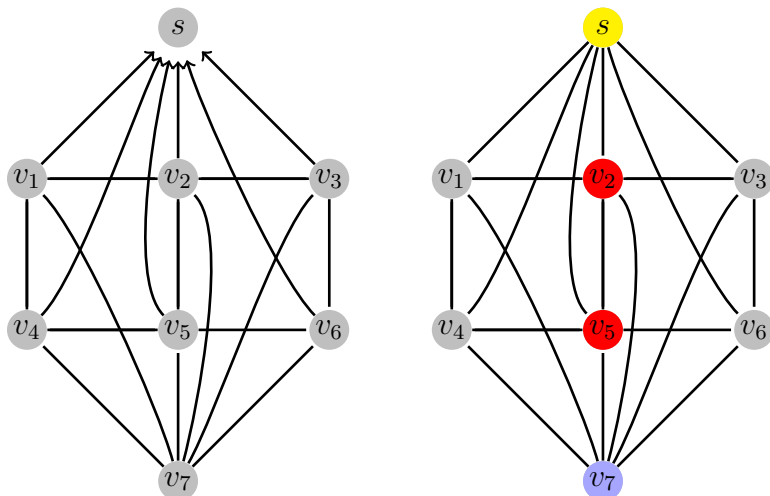


FIGURE 2. Identities for the 57×57 grid. Colors: red = 0, blue = 1, yellow = 2, pink = 3. Generated in Mathematica

3. PROPOSED APPROACH

Because of the high degree of symmetry in these identity configurations, we are motivated to look at a directed graph $\Gamma(V, E, s)$, under the group action, G the symmetries of the graph. This will partition the vertices into equivalence classes. Looking at the sandpile configurations $\sigma \in S(\Gamma)$ we call σ *symmetric* if when v_i and v_j are in the same equivalence class, $\sigma(v_i) = \sigma(v_j)$. That is, symmetric configurations have equal weights on the vertices of a given equivalence class. It is not hard to show that the symmetric configurations, $S(\Gamma)^G$, form a subgroup of the sandpile group $S(\Gamma)$.

As an alternative to looking at symmetric configurations, we might also try to “fold up” the graph along its symmetries. Consider the following example:



$$\tilde{\Delta} = \begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & -1 \\ -1 & 5 & -1 & 0 & -1 & 0 & -1 \\ 0 & -1 & 4 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 5 & -1 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 6 \end{pmatrix}.$$

Here we have identified the rows of the reduced laplacian corresponding to each equivalence class and we sum them to obtain new “rows.”

Equivalence Class	$(v_1, v_2, v_3, v_4, v_5, v_6, v_7)$
$\{v_1, v_3, v_4, v_6\}$	$(3, -2, 3, 3, -2, 3, -4)$
$\{v_2, v_5\}$	$(-1, 4, -1, -1, 4, -1, -2)$
$\{v_7\}$	$(-1, -1, -1, -1, -1, -1, 6)$

Next we select a representative column from each equivalence class to obtain a new laplacian matrix:

$$\Delta_q = \begin{pmatrix} 3 & -2 & -4 \\ -1 & 4 & -2 \\ -1 & -1 & 6 \end{pmatrix} \quad \Delta_q^T = \begin{pmatrix} 3 & -1 & -1 \\ -2 & 4 & -1 \\ -4 & -2 & 6 \end{pmatrix}$$

Let Γ_q denote the graph described by Δ_q^T . We call Γ_q the *quotient graph* and compute its sandpile group, $S(\Gamma_q)$. We would like to show that $S(\Gamma)^G$ is isomorphic to $S(\Gamma_q)$ in an attempt to fully characterize this notion of symmetric sandpiles.

We might also use algebraic geometry to approach to this problem. Again, consider a directed graph $\Gamma = (V, E, s)$, with the vertex set $V = \{v_1, \dots, v_{n+1}\}$ and let $v_{n+1} = s$ denote the sink. For each $v_i \in V$, let x_i be the corresponding indeterminate. For $i \in \{1, \dots, n+1\}$, define the polynomial

$$p_i = x_i^{\deg(v_i)} - \prod_{j: \{v_i, v_j\} \in E} x_j,$$

where E denotes the edge set of the graph and $x_{n+1} = 1$. The sandpile ideal for the graph Γ is generated these polynomials: $I_\Gamma = (p_i : i = 1, \dots, n+1) \subseteq \mathbb{C}[x_1, \dots, x_n]$. A theorem due to Cori, Rossin and Salvy states that a normal basis for $\mathbb{C}[x_1, \dots, x_n]/I_\Gamma$ with respect to the a particular monomial ordering is isomorphic to the elements of the sandpile group $S(\Gamma)$ [4]. Throughout the year we hope to develop a description of the symmetric sandpile subgroup in this context as an alternate approach question of the square in the center of the identity configuration for the $n \times n$ grid with wired boundary.

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