Abelian Sandpile Model: Symmetric Sandpiles

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Self Organized Criticality

In an equilibrium system the critical point is reached by tuning a control parameter precisely. Example: Melting water.

Definition

Self-Organized Criticality *A phenomenon of a non-equilibrium system with a critical point as attractor. Behavior of critical points same as phase transition, but without control parameters.*

Self Organized Criticality

- Sand piles exhibit static periods with intermittent sand slides.
- Avalanches are inevitable, the product of minor events in the past.

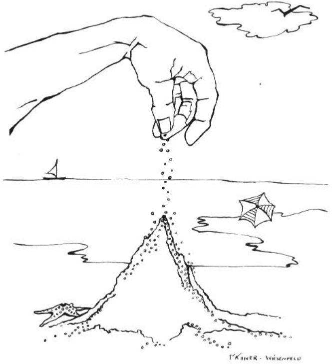


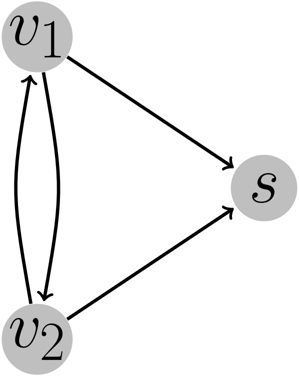
Figure: From *How Nature Works*, Per Bak.

Self Organized Criticality

- Punctuated Equilibrium (Sand avalanches correspond to cladogenesis, rapid events of branching speciation)
- Modeling Evolution (No need for meteors to explain dinosaur extinction)
- The Brain (Thought is a critical state, an avalanche triggered by visual stimulus or another thought)
- Economics (Network of consumers and producers, supply and demand avalanches. Critical economy vs. equilibrium economy)

The Model

Directed graph, $\Gamma = (V, E, s)$ (multiple edges and self loops OK).



- d_{v_1} = outdegree(v_1) = 2
- $d_s = 0$
- s is a *global sink*

The Model

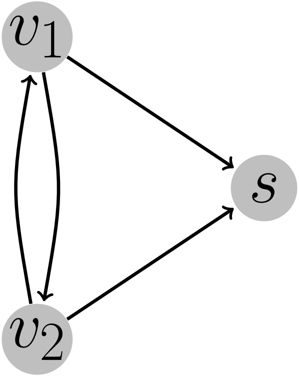
Directed graph, $\Gamma = (V, E, s)$ (multiple edges and self loops OK).

- Laplacian

$$\Delta = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

- Reduced Laplacian

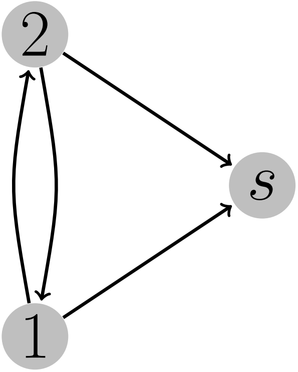
$$\tilde{\Delta} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$



Sandpiles

- Vertex set minus the sink: $\tilde{V} = V \backslash s$
- Sandpile, σ: mapping from $\tilde{V} \rightarrow \mathbb{N}$ where $\mathbb{N} = \{0, 1, 2, \ldots\}$

- $\sigma \in \mathbb{A}^2$
- $\sigma = (2, 1)$
- $\sigma(v_1) = 2$, $\sigma(v_2) = 1$
- σ *unstable* at v if $\sigma(v) \geq d_v$
- σ is *unstable* at v_1



Sand Addition: An Affine Transformation

Definition

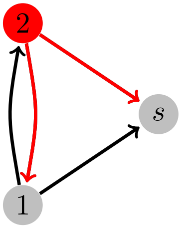
An affine transformation is a map between affine spaces and consists of a linear transformation followed by a translation: $x \mapsto Ax + b$.

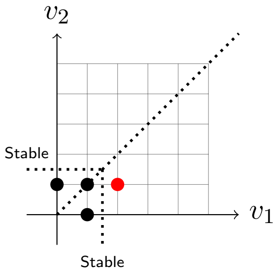
- Sandpile addition just a translation so $A = I$. Adding sand is commutative!
- $\sigma = Ix + b \geq 0$ always!

Stabilization

Firing unstable vertices: Recall the Reduced Laplacian

$$\tilde{\Delta} = \begin{pmatrix} r_1 \\ \hline r_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$





$$\sigma = I \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

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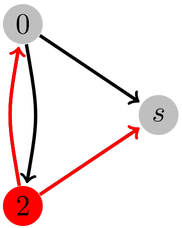
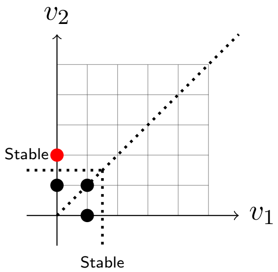


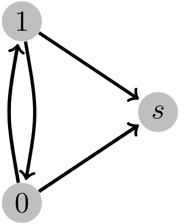


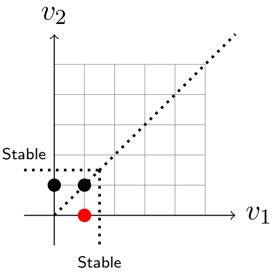
$$\sigma = I \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Stabilization

Firing unstable vertices: Recall the Reduced Laplacian

$$\tilde{\Delta} = \left(\frac{r_1}{r_2}\right) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$





$$\sigma^\circ = I\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Stabilization

Theorem

If Γ has a sink, every configuration on Γ stabilizes.

Why? The presence of a sink guarantees the the linear independence of the rows of the reduced laplacian.

Intuitively, if there is a sink, sand is leaving the graph.

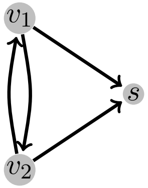
The Sandpile Group

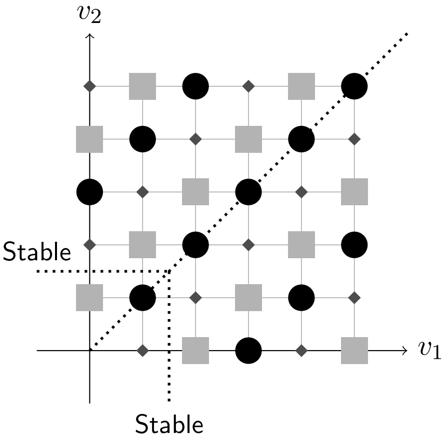
- Let $\tilde{\Delta}$ denote the reduced Laplacian of Γ, and $\tilde{V}$ the set of non-sink vertices. The Sandpile group of Γ is given by:

$$S(\Gamma) := \mathbb{Z}^{\tilde{V}} / Im\tilde{\Delta}$$

- Modding out by the integer row span of the reduced Laplacian.
- Two sandpiles are equivalent if they differ by a linear combination of rows of the reduced Laplacian.

Visualizing the Sandpile Group





Recurrent Configurations

Definition

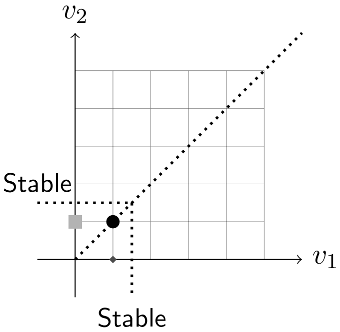
A configuration σ *is* **accessible** *if for all configurations* α*, there exists a* β *such that* $\alpha + \beta \to \sigma$*. If* σ *is stable and accessible, then* σ *is* **recurrent***.*

Simplifying the Sandpile Group

- The zero configuration is never recurrent
- Every equivalence class of $S(\Gamma)$ contains a unique stable recurrent configuration.
- Stable recurrent configurations on Γ forms an abelian group under

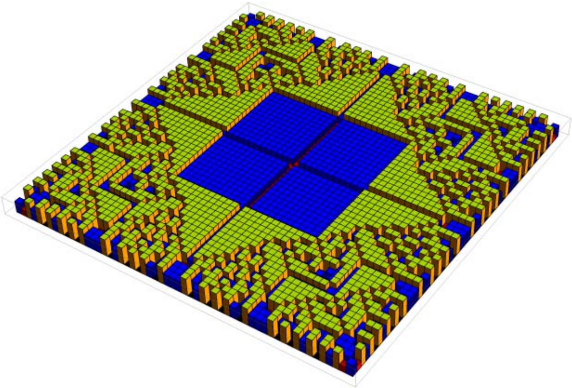
$$(\sigma, \sigma') \mapsto (\sigma + \sigma')^\circ$$

  and is isomorphic to $S(\Gamma)$.
- $|S(\Gamma)| = \det(\tilde{\Delta})$
- $I = (\sigma - \sigma^\circ)^\circ$

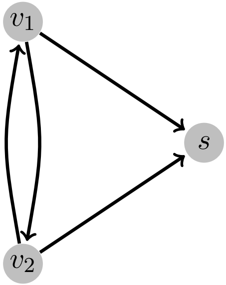


The Identity Sandpile

The Identity Sandpile: 57 by 57 Grid

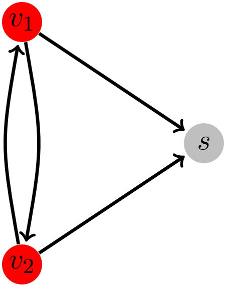


Graph Symmetry

$\Gamma =$ 

$$\tilde{\Delta} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Group Action Partitions Vertices into Equivalence Classes

G on $\Gamma =$ 

Sum the rows in each equivalence classes:

$$\tilde{\Delta} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\tilde{\Delta}^G = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Symmetric Elements

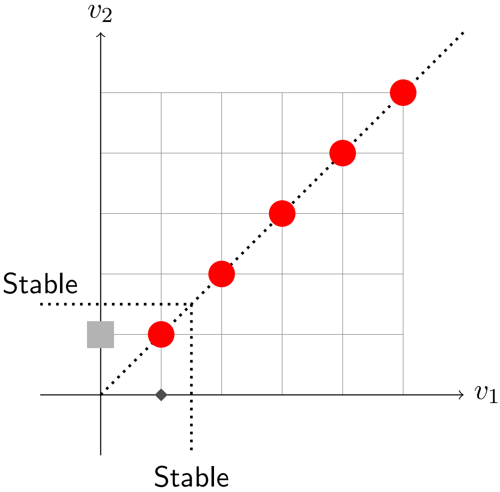
Definition

$\sigma \in S(\Gamma)$ is symmetric if it weights vertices in the same equivalence class equally. Previous example:

$$(1,1),(2,2), \textit{etc.}$$

- The symmetric elements of $S(\Gamma)$ form a subgroup, $S(\Gamma)^G$.

Symmetric Elements



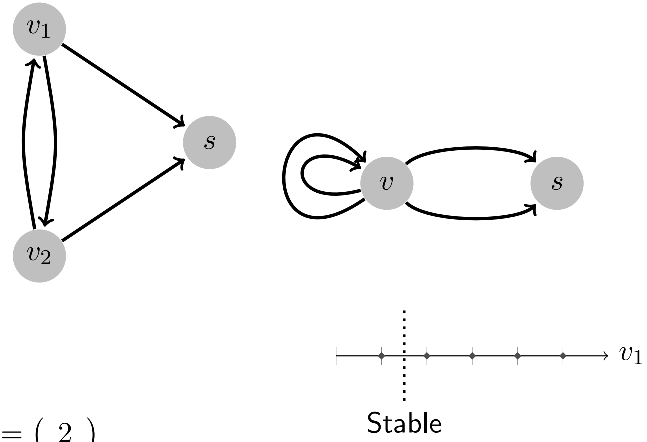
Quotient Graph

Question

Let Γ be a graph with symmetries, and let $S(\Gamma)^G$ denote its symmetric sandpile subgroup. Does there exist a graph with a sandpile group that is isomorphic to $S(\Gamma)^G$?

Hypothesized Quotient Graph

Just fold it up! $S(\Gamma/G)$



$\tilde{\Delta}(\Gamma/G) = \left(\begin{array}{c} 2 \end{array} \right)$

Thesis Exploration

1. Let $f : S(\Gamma)^G \to S(\Gamma/G)$. Is f injective?

Example

Recall: $S(\Gamma) = \{(1,1),(1,0),(0,1)\}$
$S(\Gamma)^G = \{(1,1)\}$ *and*
$S(\Gamma/G) = \{(1)\}$

2. When is it an isomorphism?
3. What relationships exist between symmetric sandpiles and domino tileings of $2n \times 2n$ grid?

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Acknowledgements and Questions

Thank you for listening!

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- Second Reader: Professor Perkinson (Reed College)