PCMI | **Pizza & Problem Solving # 1** | Andrew J. Bernoff
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Hidden Symmetries

This problem session is modelled after the HMC Putnam Preparation Problem Solving Seminar co-led with Francis Su. Additional resources (and this problem set) can be found at:

`http://www.math.hmc.edu/~ajb/PCMI/problem_solve.html`

A1: (a) A deck of fifty-two playing cards and one joker is shuffled and cards are turned over from the top one at a time until the joker appears. On average, how many cards are dealt before the joker appears?

(b) Suppose there are N jokers instead of one. Now, on average, how many cards are dealt before the joker appears?

A2: (a) Determine the shortest curve in an equilateral triangle which divides it into two regions of equal area.

(b) Determine the shortest curve in a square which divides it into two regions of equal area.

A3: (a) Evaluate

$$\int_0^{\pi/2} \frac{\mathrm{d}x}{1+(\tan(x))^{\sqrt{2}}}.$$

(Putnam, 1980)

(b) Evaluate

$$\int_0^1 2^{x^2} - 1 + \sqrt{\log_2(x+1)}\ \mathrm{d}x.$$

A4: (a) Given an isosceles right triangle with legs of length 1, suppose a bug wants to travel from the midpoint of one leg to the vertex at the right angle, but must touch the hypotenuse somewhere in between. What is the shortest distance the bug can travel?

(b) Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the x-axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists. (Putnam, 1998)

And for a little bit of variety...

A5: Picture Hanging: Two nails are put into a wall at the same height but a few inches apart. A picture is to be hung with a piece of wire that is attached to its two upper corners and looped around the nails. Show how to wrap the wire so that if either nail is removed the picture will fall to the ground. (Winkler)

Hints:

1. Can you find a"circular" symmetry in this problem?

2. a) It helps to know that you can arrange six equilateral triangles into a hexagon. b) The answer for this one is easy, but the proof takes some work.

3. Both of these problems have "calculus-free" solutions a) Can you find the symmetry in the integrand? b) In this problem, the two pieces of the integrand can be combined geometrically.

4. Both these problems can be simplified by reflection symmetries.

5. For the topologist among you, thinking about "winding numbers".