PCMI | **Pizza & Problem Solving # 2** | Andrew J. Bernoff
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Lattice Points and Polygons

This problem session is modelled after the HMC Putnam Preparation Problem Solving Seminar co-led with Francis Su. Additional resources (and this problem set) can be found at:

`http://www.math.hmc.edu/~ajb/PCMI/problem_solve.html`

B1: (a) An explorer lives on the planet Qubit which is in the shape of a perfect cube with a side of length q. She wishes to walk from the north pole (which is a vertex of the cube) to the south pole (which is the opposite vertex). What is the minimum distance she needs to walk?

(b) The moon of Qubit is a rectangular solid with sides s, $2s$ and $3s$. What is the minimum distance one needs to walk to get from one vertex of the moon to the vertex opposite it?

B2: Suppose you have a polygon of perimeter 12, whose vertices are all on lattice points and whose sides all have integer lengths. Show that its area can be $3, 4, 5, 6, 7, 8$ or 9. (Inspired by Wagon)

B3: Show there is no equilateral triangle whose vertices are all lattice points.

B4: (a) Prove that any pentagon whose vertices are lattice points must have an area greater than or equal to $3/2$.

(b) Show that among any five lattice points in the plane that there are a pair whose midpoint is a lattice point also.

(c) Prove that any convex pentagon whose vertices (no three of which are collinear) are lattice points must have area greater than or equal to $5/2$. (Putnam 1990)

B5: Suppose a cube has vertices that are lattice points. Show that the length of its side must be an integer. (Gelca & Andreescu)

And for a little bit of variety...

B5: Greedy Pirates: You have 1000 pirates, who are all extremely greedy, heartless, and perfectly rational. They're also aware that all the other pirates share these characteristics. They're all ranked by the order in which they joined the group, from pirate one down to pirate one thousand.

They've stumbled across a huge horde of treasure, and they have to decide how to split it up. Every day they will vote to either kill the lowest ranking pirate, or split the treasure up among the surviving pirates. If 50% or more of them vote to split it, the treasure gets split. Otherwise, they kill the lowest ranking pirate and repeat the process until half or more of the pirates decide to split the treasure.

The question, of course, is at what point will the treasure be split, and what will the precise vote be?

(Newheiser & Wu)

Hints:

1. Think about unfolding a map of the planet. Why does the shortest path between the vertices look like a line segment between two lattice points?
2. Start with rectangles and triangles with integer length sides
3. Can you remember/derive the formula for the area of an equilateral triangle in terms of its side length?

4.(b) Think about the parity (even/odd) of the coordinates of the vertices. (c) Use part (b).

5. For this one you need the triple product formula for the volume of the cube.
6. Try some low number examples and see if you can use induction to find the answer.