PCMI | **Pizza & Problem Solving # 1** | Andrew J. Bernoff
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Induction and Recursion

This problem session is modelled after the HMC Putnam Preparation Problem Solving Seminar. Additional resources (and this problem set) can be found at:

`http://www.math.hmc.edu/~ajb/PCMI/problem_solve.html`

A1: The Fibonacci numbers are defined by the two-term recurrence relationship

$$F_1 = 1 \qquad F_2 = 1 \qquad F_{n+2} = F_{n+1} + F_n \quad \text{for} \quad n = 1, 2, 3, \ldots$$

Show

(a) $F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$

(b) $F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$

(c) $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$

A2: Show that every number in the sequence

$$1007, 10017, 100117, 1001117, \ldots$$

is divisible by 53. (Engel)

A3: The sequence $G_n = F_{2n}$ consists of every other Fibonacci number; that is $G_1 = F_2 = 1$, $G_2 = F_4 = 3$ and so forth. Show that G_n satisfies a linear recurrence of the form

$$G_n = aG_{n-1} + bG_{n-2}$$

where a and b are constants to be determined. (Vakil)

A4: Let S_n be the number of subsets of $\{1, 2, \ldots, n\}$ that contain no two consecutive elements of $\{1, 2, \ldots, n\}$. So, for example, if $n = 2$, then $\{1\}$, $\{2\}$ and the empty set, $\{\}$, are acceptable but $\{1, 2\}$ is not, so $S_2 = 3$. Determine S_n.

A5: The mathematician Edouard Zeckendorf explored writing positive integers as sums of distinct Fibonacci numbers. For example:

$$1 = F_1 \qquad 28 = 2 + 5 + 21 = F_2 + F_5 + F_8 \qquad 100 = 3 + 8 + 89 = F_4 + F_6 + F_{11}$$

(a) Show that every number can be written as a sum of distinct Fibonacci numbers.

(b) Show that every number can be written as a sum of distinct, non-consecutive Fibonacci numbers.

And for a little bit of variety...

A6: Not quite origami . . . Cut the central square out of a 5×5 grid of 25 squares. Can you cut the resulting shape into two pieces that can be arranged, by various folds, into the surface of a $2 \times 2 \times 2$ cube? (Quantum Magazine)

Hints:

1. These identities can be shown by induction. What is the base case? What is the induction hypothesis?
2. Can you find a recursion relationship for this sequence? How does this help?
3. You can solve for a and b from two examples. How can you prove the result?
4. Compute S_n for a few examples. Do you see a pattern?
5. The greedy algorithm works here.