Putnam TNG #2 – Evaluation of Integrals

The Putnam TNG seminar assumes that you have tried to work the problems in advance of the seminar and have solved at least one problem that you are willing to present in class.

These problems will be discussed on 9/24/02.

B1: (a) Evaluate the integrals

$$I_n = \int_0^{\pi} \frac{\sin(nx)}{\sin(x)}\, \mathrm{d}x \quad \text{and} \quad J_n = \int_0^{\pi} \left(\frac{\sin(nx)}{\sin(x)}\right)^2 dx$$

for $n = 1, 2, 3 \ldots$. (Hardy & Williams)

(b) Evaluate the integrals

$$I_n = \int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^{-x})\sin(x)}\, dx$$

for $n = 1, 2, 3 \ldots$. (IMC 1996)

B2: (a) Evaluate the integral

$$I(a,b) = \int_0^{\infty} \frac{e^{ax} - e^{bx}}{x(e^{ax}+1)(e^{bx}+1)}\, \mathrm{d}x$$

for $a > b > 0$. (Hardy & Williams)

(b) Evaluate

$$\int_0^{\infty} \frac{Arctan(\pi x) - Arctan(x)}{x}\, dx\ .$$

(Putnam 1982)

B3: Let H be the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$, C the unit circle $\{(x, y, 0) : x^2 + y^2 = 1\}$, and P the regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A \sin \alpha + B \cos \beta$, where A, B, α, β are real numbers. (Putnam 1998)

B4: Let $||u||$ denote the distance from the real number u to the nearest integer (for example $||2.8|| = .2 = ||3.2||$). For positive integers n, let

$$a_n = \frac{1}{n}\int_1^n \left|\left|\frac{n}{x}\right|\right| \, dx$$

Determine $\lim_{n\to\infty} a_n$. You may assume the identity

$$\frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\frac{8}{7}\frac{8}{9}\cdots = \frac{\pi}{2}\ .$$

(Putnam 1983)

B5: Let

$$F(x) = x^4 e^{-x^3}\int_0^x \int_0^{x-u} e^{u^3+v^3} \, dv \, du$$

Find $\lim_{x\to\infty} F(x)$ or prove that it doesn't exist (Putnam, 1983)

B6: Let

$$I_n = \int_0^\pi \frac{e^x}{1+\cos^2(nx)} \, \mathrm{d}x$$

for $n = 1, 2, 3 \ldots$. Evaluate $\lim_{n\to\infty} I_n$. (Bernoff)

B7: Let $f(x)$ be a continuous function on the interval $I = [0, 1]$ with the property

$$xf(y) + yf(x) \leq 1$$

for x, y in I. Prove that

$$\int_0^1 f(x) \, dx \leq \frac{\pi}{4}$$

and find a function $f(x)$ for which equality is obtained. (IMC 1997)