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Problem 10762 [1999,864]. Let $x_1 = 1$ and for $m \geq 1$ let $x_{m+1} = (m+3/2)^{-1} \sum_{k=1}^{m} x_k x_{m+1-k}$. Evaluate $\lim_{m\to\infty} x_m/x_{m+1}$. *Solution by Andrew Bernoff and Karl Mahlburg (student), Harvey Mudd College, Claremont, CA.* Define a generating function $f(z) = \sum_{n=1}^{\infty} x_n z^n$. Multiplying both sides of the given sum by $(m + 3/2)z^{m+1}$ and summing over positive m leads to a DE for $f(z)$,

$$z f_z + \frac{1}{2} f - \frac{3}{2} z = f^2 \ .$$

Our generating function is a solution to this DE which is analytic in a neighborhood of the origin, where $f(0) = 0$.

Making the substitution $s = \sqrt{z}$, $f(z) = \frac{1}{2} + s g(s)$ yields a separable DE, $g_s = 2g^2 + 3$. Selecting the appropriate solution for g yields the solution for f,

$$f(z) = \frac{1}{2} \left[1 - \sqrt{6z} \cot(\sqrt{6z}) \right]$$

which is meromorphic, has a removeable singularity at the origin and poles at $N\pi/\sqrt{6}$ for N a non-zero integer. Noting that x_m is positive, we see that the required limit is the radius of convergence of the power series for $f(z)$ at $z = 0$, given by the distance to the nearest pole in the complex plane, that is $\lim_{m\to\infty} x_m/x_{m+1} = \pi/\sqrt{6}$.