

Induction and Deduction

This is a sample problem set from the Harvey Mudd College Putnam Problem Solving Seminar, which runs every Tuesday night in the fall semester in preparation for the annual Putnam Mathematics Competition.

A1: Show that

$$1^2 + 3^2 + 5^2 + 7^2 + \cdots + (2n - 1)^2 = \frac{n(4n^2 - 1)}{3}$$

A2: The Fibonacci numbers are defined by the recurrence relationship

$$F_1 = 1 \quad F_2 = 1 \quad F_{n+2} = F_{n+1} + F_n \quad \text{for } n = 1, 2, 3, \dots$$

Show

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$$

(Lozansky & Rousseau)

A3: Show that every number in the sequence

$$1007, 10017, 100117, 1001117, \dots$$

is divisible by 53.

(Engel)

A4: Show that for $n \geq 6$ a square can be dissected into n smaller squares, not necessarily all of the same size.

A5: You have coins C_1, C_2, \dots, C_n . For each k , C_k is biased so that, when tossed, it has probability $1/(2k + 1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n . (Putnam 2001)

And now for something completely different . . .

A6: An absent-minded Professor goes out for a walk with his trusty compass. He checks his compass to determine North, walks for 10 miles in a straight line, and then repeats the process walking 10 miles East and 10 mile South. Curiously, he ends up exactly where he started. Where did the Professor start his journey ?

Hints:

1. Subtracting the equality for $n = p$ from $n = p + 1$ may give you a hint.
2. Use the same idea as the first problem, together with the recursion relationship.
3. Can you write down a recursion relationship for this sequence?
4. First find a solution for $n = 6, 7, 8$. Then do induction on $n + 3$.
5. Write down a recursion relationship and work out the answer for the first few values of n . Can you guess the answer and show it is correct by induction?
6. I hope the Professor was wearing snowshoes.