Liouville Properties for Nonsymmetric Diffusion Operators

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We consider nonsymmetric diffusion operators on complete noncompact Riemannian manifolds.

$$L = \Delta + X$$

$\Delta =$ the Laplace-Beltrami operator
$X =$ is a vector field.

We analyze positive solutions of $Lu = 0$

We establish

- Gradient estimates;
- Harnack inequalities;
- Maximum Principles; and
- Liouville properties

under

- Mild lower bounds on the Ricci curvature; and

- Certain conditions on the vector field.

- **Strong Liouville Property**

  $$Lu = 0 \qquad \text{and} \quad u > 0 \quad \Longrightarrow \quad u \equiv C$$

- **Weak Liouville Property**

  $$Lu = 0 \qquad \text{and} \quad u\text{−bounded} \quad \Longrightarrow \quad u \equiv C$$

- L^p **Liouville Property**

  $$Lu = 0 \qquad \text{and} \quad u \in L^p \quad \Longrightarrow \quad u \equiv C$$

Strong Liouville Property

Theorem 1. *(Castañeda 98) Let* (M, g) *be a complete, connected, noncompact Riemannian manifold of dimension* $m \geq 2$*, such that*

$$\mathrm{Ric}(x) \geq -f(r(x))g,$$

where $r(x)$ *denotes the distance from* x *to a fixed point* $p \in M$*, and* f *is a positive increasing function such that* $\lim_{r\to\infty} \dfrac{f(r)}{r^2} = 0$.

Let X *be a vector field on* M *satisfying*

(i) $\mathrm{Ric}(Z, Z) - D_Z X \cdot Z \geq 0$ for all $Z \in TM$.

(ii) $|X_q| \to 0$ *as* $q \to \infty$.

Then the operator $L = \Delta + X$ *satisfies the strong Liouville property.*

Condition (ii) is necessary

$$u(x^1, \ldots, x^n) = e^{x^1} \qquad \text{in } \mathbf{R}^n$$

$$\Delta u + X \cdot \nabla u = 0$$

for $X = -\nabla(x^1)$.

Brief History

- $M = \mathbf{R}^m$ = well known theorem.

- $L = \Delta$, Ric ≥ 0 = Famous Yau's theorem of 1975.

- (Gidas - Spruck 1981) $L = \Delta + X$, Ric ≥ 0. Extra conditions on the decay of X and of DX to zero. They use integral methods.

- (Li Jiayu 1991) Same lower bound on Ricci curvature. Concavity condition

$$\text{Ric}(Z, Z) - D_Z X \cdot Z > c|Z|^2 \qquad c > 0.$$

- (Xiang - Dong Li 2005) $L = \Delta + \nabla\Phi$. So $X = \nabla\Phi$. Concavity condition

$$\text{Ric}(Z, Z) - D_Z X \cdot Z - \alpha|\langle X, Z\rangle|^2 \geq 0 \qquad \alpha > 0.$$

Positive Ricci Curvature

Corollary 2. *Let M be a complete noncompact Riemannian manifold of strictly positive Ricci curvature such that* $\mathrm{Ric}(x) \to 0$ *as* $x \to \infty$.

Let X be a vector field such that

$$|D_Z X \cdot Z| + |X| \leq C\,\mathrm{Ric}(Z, Z)$$

for some positive constant C, and all unit tangent vectors Z.

Then,

$L = \Delta + \varepsilon X$ *satisfy the strong Liouville property as long as* $|\varepsilon| \leq \frac{1}{C}$.

The Weitzenböck Formula

$$\Delta|\nabla u|^2 = 2\Big[|\mathcal{H}u|^2 + \mathrm{Ric}(\nabla u, \nabla u) + \nabla(\Delta u)\cdot\nabla u\Big]$$

$$\begin{aligned} \mathrm{Ric}_X &:= \mathrm{Ric} - DX \\ \mathrm{Ric}_{X,\alpha} &:= \mathrm{Ric} - DX - \alpha X \otimes X \end{aligned}$$

The Weitzenböck Formula for L (Bakry-Emery)

$$L|\nabla u|^2 = 2\Big[|\mathcal{H}u|^2 + \mathrm{Ric}_X(\nabla u, \nabla u) + \nabla(Lu)\cdot\nabla u\Big]$$

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Proof. We observe that

$$X|\nabla u|^2 = 2\langle D_X \nabla u, \nabla u\rangle = 2\langle X, D_{\nabla u}\nabla u\rangle \quad (1)$$

and that

$$\begin{aligned} \langle \nabla(Xu), \nabla u\rangle &= \nabla u(Xu) = \nabla u\langle X, \nabla u\rangle \qquad (2) \\ &= \langle D_{\nabla u}X, \nabla u\rangle + \langle X, D_{\nabla u}\nabla u\rangle. \end{aligned}$$

The Bakry-Emery Weitzenböck formula is obtained by adding Equation (1) to the Weitzenböck formula in view of (2). □

Laplacian Comparison Theorem

Lemma 3. *(Generalized Laplacian comparison theorem) Let* M^n *be a complete Riemannian manifold and* X *a smooth vector field on* M *such that for some positive number* α *and some* $K \geq 0$, $\mathrm{Ric}_{\alpha,X} \geq -K$. *Let* $\rho(x)$ *denote the distance from* x *to a fixed point* $p \in M$. *Then there exists a positive constant* $C = C(n, \alpha)$ *such that*

$$L\rho \leq C\frac{(1+\sqrt{K}\rho)}{\rho.} \tag{3}$$

$$L|\nabla u|^2 = 2\Big[|\mathcal{H}u|^2 + \mathrm{Ric}_X(\nabla u, \nabla u) + \nabla(Lu)\cdot\nabla u\Big]$$

Apply to $u = \rho =$ distance to a fixed point.

$$0 = |\mathcal{H}\rho|^2 + \mathrm{Ric}_X(\nabla\rho, \nabla\rho) + \nabla(L\rho)\cdot\nabla\rho\Big]$$

Along any geodesic integral curve of $\nabla\rho$:

$$0 \geq \frac{(\Delta\rho)^2}{n-1} + \alpha|X\rho|^2 - K + (L\rho)'$$

$$0 \geq C(\alpha, n)(L\rho)^2 - K + (L\rho)'$$

$\Longrightarrow$ Upper bound for $L\rho$.

Cheng - Yau Gradient Estimate

$$u > 0, \qquad \Delta u = 0 \qquad \text{in } B_p(R)$$

$$\text{Ric} \geq 0$$

Then, there exists a constant $C = C(n)$ such that

$$\frac{|\nabla u|}{u}(p) \leq \frac{C}{R}$$

Conjecture

Best C is n.

Gradient estimate without concavity condition

Theorem 4. *Let M be a complete noncompact Riemannian manifold, and X an arbitrary vector field on M. Let u be a positive solution of the equation $\Delta u + X \cdot \nabla u = 0$ in a geodesic ball $B_p(R)$ on which $\text{Ric} \geq -(n-1)k$.*

Then there exists a constant $C = C(n)$ such that

$$\frac{|\nabla u|}{u}(p) \leq C\Big[\ \frac{1}{R} + \sqrt{k} + |X|_{1,B_p(R)}\ \Big]$$

where

$$|X|_{1,B_p(R)} = \sup\Big[|X_q|^2 + |D_{Z_q}X|\Big]^{\frac{1}{2}}$$

and the sup *is taken over all $q \in B_p(R)$ and all unit vectors $Z_q \in B_p(R)$.*

With concavity condition

Lemma 5. *Let* u *be a positive solution of the equation* $\Delta u + X \cdot \nabla u = 0$ *in a geodesic ball* $B_p(R)$ *on which* $\mathrm{Ric} \geq -(n-1)k$.

Assume further that the vector field X *is concave.*

Then there exists a constant $C = C(n)$ *such that*

$$\frac{|\nabla u|^2}{u^2}(p) \leq C\Big[\ \frac{1}{R^2} + \frac{\sqrt{k}}{R} + |X|_{B_p(R)}\ \Big], \qquad (4)$$

where $|X|_{B_p(R)} = \mathrm{sup}_{q \in B_p(R)} |X|_q$.

Yau's Gradient Estimate

$$u > 0, \qquad \Delta u = 0 \qquad \text{on } M$$

$$\text{Ric} \geq -K$$

Then, there exists a constant $C = C(n)$ such that

$$|\nabla u| \leq C\sqrt{K}$$

Theorem 6. *(Xiang- Dong Li) Let M be a complete noncompact Riemannian manifold and X a vector field on M such that*

$$\mathrm{Ric}_{X,\alpha} \geq -K$$

for some constants $\alpha > 0$ and $K \leq 0$.

Let u be a positive solution of $(\Delta + X)u = 0$. Then

$$|\nabla u| \leq C\sqrt{K}u$$

where $C = \sqrt{(\frac{1}{\alpha} + n - 1)}$.

Lemma 7. *Let M be a complete noncompact Riemannian manifold, and $r(x)$ the distance from x to a fixed point $p \in M$. Let* Ric $\geq -(n-1)k$ *on the geodesic ball $B_p(R)$. Then there exists a smooth function ξ defined on a neighborhood of $\overline{B_p(R)}$ such that*

(i) $\xi(p) = R^2$ *and* $\xi \leq 0$ *on* $\partial B_p(R)$;

(ii) $|\nabla\xi|^2 \leq CR^2$ *whenever* $\xi > 0$;

(iii) $-\xi\Delta\xi \leq C\left[\sqrt{k}R^3 + R^2\right]$ *whenever* $\xi > 0$,

where C is a constant which depends only on n.

Lower bound on the Hessian

Theorem 8. *Let M be any Riemannian manifold and u a smooth function defined on M. Then for every $\varepsilon > 0$, the following inequality holds in the set of noncritical points of u.*

$$\begin{aligned} |\mathcal{H}u|^2 &\geq |\nabla(|\nabla u|)|^2 \\ &+ \frac{1}{n-1}\Big[|\nabla(|\nabla u|)|^2(1-\varepsilon) + |\Delta u|^2(1-\frac{1}{\varepsilon})\Big] \end{aligned}$$

Proof. Let $|\nabla u|(p) \neq 0$ and consider a local orthonormal frame $e_1, \ldots, e_n$ around p such that $e_1 = \dfrac{\nabla u}{|\nabla u|}$. In particular, $|\nabla u| = e_1(u) = u_1$.

Let u_{ij} be the components of $\mathcal{H}u$ relative to this orthonormal frame. A simple computation shows that $u_{1i} = e_i(u_1)$, and therefore $|\nabla(|\nabla u|)|^2 = \sum_{i=1}^n u_{1i}^2$.

Now,

$$\begin{aligned} |\mathcal{H}u|^2 - |\nabla|\nabla u||^2 &= \sum_{i,j} u_{ij}^2 - \sum_{j=1}^n u_{j1}^2 \\ &= \sum_{j=2}^n u_{1j}^2 + \sum_{i\geq 2, j\geq 2} u_{ij}^2 \end{aligned}$$

and

$$\sum_{i\geq 2, j\geq 2} u_{ij}^2 \geq \frac{(\sum_{j=2}^n u_{jj})^2}{n-1} = \frac{(\Delta u - u_{11})^2}{n-1}. \quad (5)$$

Therefore, for any $\varepsilon > 0$,

$$|\mathcal{H}u|^2 - |\nabla|\nabla u||^2 \geq$$

$$\textstyle\sum_{j=2}^{n} u_{1j}^2 + \frac{1}{n-1}\left[(\Delta u)^2(1-\frac{1}{\varepsilon}) + u_{11}^2(1-\varepsilon)\right]$$

$$\geq \tfrac{1}{n-1}\left[(\Delta u)^2(1-\tfrac{1}{\varepsilon}) + |\nabla|\nabla u||^2(1-\varepsilon)\right]$$

$\square$

Maximum Principles

Theorem 9. *Let Ω be an open subset of a Riemannian manifold M, and X a smooth concave vector field on Ω.*

Then for any solution of $(\Delta + X)u = 0$ in Ω one of the following two conditions holds:

(1) $|\nabla u|(x) < \sup_{\Omega} |\nabla u| \quad \forall x \in \Omega$; *or else*

(2) ∇u *is a parallel vector field on* Ω, u *is a harmonic function, and the domain* Ω *is locally isometric to* $\mathbf{R}\times N$ *for some manifold* N.

The logarithm of u satisfies same type of equation

Proposition 10.

$$u > 0, \qquad \Delta u + X \cdot \nabla u = 0.$$

Then the function $v = \log u$ *satisfies*

$$\Delta v + Z \cdot \nabla v = 0,$$

with $Z = X + \nabla(\log u)$.

Max. Principle for $\dfrac{|\nabla u|}{u}$

Theorem 11. *Let Ω be a connected open set in a Riemannian manifold M, and X a smooth concave vector field X on Ω.*

Let also u be a nonconstant positive solution of $(\Delta + X)u = 0$.

Then either

$$\frac{|\nabla u|}{u}(x) < \sup_{\Omega} \frac{|\nabla u|}{u} \quad \textit{for every } x \in \Omega,$$

or else, $\nabla(\log u)$ is a parallel vector field; the domain Ω is locally isometric to $\mathbf{R}\times N$, where N is a level surface of u; and $u(x) = ae^{bt}$ for some constants a, b, where t is the distance from x to a fixed level surface of u.

Proof. We apply the Weitzenböck formula to the function $v = \log u$:

$$\Delta(|\nabla v|^2) = 2\Big[|\mathcal{H}v|^2 + \mathrm{Ric}(\nabla v, \nabla v) + \nabla v(\Delta v) \cdot \nabla v\Big]$$

$$= 2\Big[|\mathcal{H}v|^2 + \mathrm{Ric}(\nabla v, \nabla v) - \nabla v(X + \nabla v) \cdot \nabla v\Big]$$

$$= 2\Big[|\mathcal{H}v|^2 + \mathrm{Ric}(\nabla v, \nabla v)$$

$$-D_{\nabla v}X \cdot \nabla v - D_{\nabla v}\nabla v \cdot \nabla v - (X + \nabla v) \cdot D_{\nabla v}\nabla v\Big],$$

where the second equation follows from Proposition 10 and the third equation from properties of covariant derivatives.

Now taking into account the $2D_{\nabla v}\nabla v = \nabla|\nabla v|^2$ and the concavity condition on X, we obtain,

$$\Delta(|\nabla v|^2) + (X + 2\nabla v) \cdot \nabla|\nabla v|^2 \geq 0.$$

By Hopf's maximum principle, $|\nabla(\log u)|$ has no interior local maxima unless $|\nabla(\log u)| \equiv C$ for some constant C. One now completes the proof as in Theorem 9. □

Proof of the strong Liouville property

Proof. Consider a point $p \in M$. Given any $\varepsilon > 0$ we can find an $R > 0$ such that $|X|(x) < \varepsilon$ whenever the distance for x to p is greater than R.

Then by the gradient estimate (4) and the lower bound on the Ricci curvature,

$$\frac{|\nabla u|^2}{u^2}(q) \leq C\Big[\frac{1}{R^2} + \frac{\sqrt{f(3R)}}{R} + \varepsilon\Big], \qquad (6)$$

for all $q \in \partial B_p(2R)$.

Since by hypothesis, $\lim_{R\to\infty} \frac{f(R)}{R^2} = 0$, we conclude that $\lim_{q\to\infty} \frac{|\nabla u|}{u}(q) = 0$.

Theorem 11 implies that $\frac{|\nabla u|}{u}(p) = 0$.

Therefore, the function u is a constant. □

Maximum principles involving a Payne function

Proposition 12. *Let* Ω *be an open subset of a Riemannian manifold* M, *and* X *any vector field on* Ω. *Let* u *be a solution of the equation*

$$\Delta u + X \cdot \nabla u + f(u) = 0 \quad \text{in } \Omega, \tag{7}$$

where f *is an arbitrary differentiable real valued function. Let* $Y = 2f\nabla u - \frac{1}{2}\nabla P$, *where* $P = |\nabla u|^2 + 2\int_0^u f(s)\,ds$. *Then*

$$\frac{Y \cdot \nabla P}{|\nabla u|^2} = 2\Big[f^2 - |\nabla|\nabla u||^2\Big] \tag{8}$$

in the set of noncritical points of u.

Proof. The following straightforward computations lead to the claim.

$$\begin{aligned} Y \cdot \nabla P &= (2f\nabla u - \frac{1}{2}\nabla P) \cdot (\nabla(|\nabla u|^2) + 2f(u)\nabla u) \\ &= (2f\nabla u - \frac{1}{2}\nabla(|\nabla u|^2) \\ &\quad -f(u)\nabla u) \cdot (\nabla(|\nabla u|^2) + 2f(u)\nabla u) \\ &= \frac{1}{2}(2f\nabla u - \nabla|\nabla u|^2) \cdot (\nabla|\nabla u|^2 + 2f(u)\nabla u) \\ &= \frac{1}{2}\Big[4f^2|\nabla u|^2 - |\nabla|\nabla u|^2|^2\Big] \\ &= 2f^2|\nabla u|^2 - 2|\nabla u|^2|\nabla|\nabla u||^2 \end{aligned}$$

□

The following is a generalization of a theorem of Payne and Stakgold to Riemannian manifolds.

Theorem 13. *Let Ω be an open connected subset of a Riemannian manifold M, and X a concave vector field on Ω. Let u be a solution of the equation*

$$\Delta u + X \cdot \nabla u + f(u) = 0 \quad \text{in } \Omega, \qquad (9)$$

where f is an arbitrary differentiable real valued function. Then the function $P = |\nabla u|^2 + 2\int_0^u f(s)\,ds$ cannot attain a local maximum at a point where $|\nabla u| \neq 0$ unless $P \equiv C$ for some constant C.

Proof. Easy computations show that $\nabla P = \nabla|\nabla u|^2 + 2f(u)\nabla u$, and $\nabla u(\Delta u) = -D_{\nabla u}X \cdot \nabla u - X \cdot D_{\nabla u}\nabla u$. Hence, using the Weitzenböck formula we obtain

$$\Delta P =$$

$$2\Big[|\mathcal{H}u|^2 + Ric(\nabla u, \nabla u) + \nabla u(\Delta u) + f(u)\Delta u + f'(u)|\nabla u|^2\Big]$$

$$= 2\Big[|\mathcal{H}u|^2 + Ric(\nabla u, \nabla u) - D_{\nabla u}X \cdot \nabla u -$$

$$X \cdot D_{\nabla u}\nabla u - f(u)X \cdot \nabla u - f(u)^2\Big]$$

On the other hand,

$$X \cdot \nabla P = 2X \cdot D_{\nabla u}\nabla u + 2f(u)X \cdot \nabla u. \quad (10)$$

Therefore,

$$\begin{aligned}\Delta P + X\cdot\nabla P &= 2\Big[|\mathcal{H}u|^2 \\ +Ric(\nabla u,\nabla u) &- D_{\nabla u}X\cdot\nabla u - f(u)^2\Big] \quad (11)\end{aligned}$$

Adding together the equations (11) and (8) we obtain

$$\Delta P + X\cdot\nabla P + \frac{Y\cdot\nabla P}{|\nabla u|^2}$$

$$= 2\Big[|\mathcal{H}u|^2 - |\nabla|\nabla u||^2 + Ric(\nabla u,\nabla u) - D_{\nabla u}X\cdot\nabla u\Big] \geq 0.$$

By Hopf's maximum principle, P cannot attain a maximum at a point where $\nabla u \neq 0$, unless $P \equiv C$ for some constant C.

$\square$

Theorem 14. *Let* Ω *be an open connected subset of a Riemannian manifold* M. *Let* X *be a vector field on* Ω *and* $f : \mathbf{R} \to \mathbf{R}$ *a differentiable real valued function such that*

(i) $u = 0$ on $\partial\Omega$;

(ii) $f(0) = 0$;

(iii) $u > 0$ in Ω;

(iv) X *is concave, i.e.,* $Ric - DX \geq 0$; *and*

(v) $X \cdot \eta \geq -h,$ *where* η *is outer unit normal vector field, and* h *is the mean curvature of* $\partial\Omega$.

Then,

(1) The function $P = |\nabla u|^2 + 2\int_0^u f(s)\,ds$ cannot attain the maximum on the boundary of Ω.

(2) $P \leq 2\int_0^{u_{max}} f(s)\,ds = F(u_{max})$

(3) $|\nabla u|^2 \leq F(u_{max}) - F(u)$,

where $F(t) := 2\int_0^t f(s)\,ds$.

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