EXISTENCE OF RADIAL SOLUTIONS FOR AN ASYMPTOTICALLY LINEAR p-LAPLACIAN PROBLEM

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1. The problem

The problem is

$$\begin{cases} \Delta_p u + g(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\mathrm{D}_p)$$

where

$\Omega = \{x \in \mathbb{R}^N : \|x\| < 1\} \subset \mathbb{R}^N$, $N \geq 2$, $g \in C^1(\mathbb{R})$, $g(0) = 0$ and Δ_p is the p-Laplacian operator given by

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \quad p > 1.$$

Let $\varphi_p(t) := |t|^{p-2}t$ for $t \neq 0$, $\varphi_p(0) = 0$ and let

$$0 < \mu_1(p) < \mu_2(p) < \cdots < \mu_k(p) < \mu_{k+1}(p) < \cdots$$

be the sequence of radial eigenvalues of $-\Delta_p$.

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The nonlinearity g satisfies:

(g_1) There are positive constants β_1, β_2 such that

$g(\beta_1) = 0$ and

$$\beta_2 = \inf\{t \in [\beta_1, \infty) : \forall\, s > t,\ g(s) > 0\}.$$

(g_2) $\exists j \in \mathbb{N},\ \mu_j(p) < \lim\limits_{|t| \to \infty} \dfrac{g(t)}{\varphi_p(t)} =: \lambda_\infty < \infty;$

(g_3) $\lim\limits_{t \to 0} \dfrac{g'(t)}{\varphi_p'(t)} = \lambda_\infty;$

(g_4) there exists $C > 0$ such that

$$\limsup_{t \to \beta_i} \left| \frac{g(t)}{\varphi_p(t - \beta_i)} \right| \leq C \quad \text{for} \quad i = 1, 2.$$

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined as

$$f(t) := g(t) - \lambda_\infty \varphi_p(t).$$

This function f inherits from g the following properties:

(f_0) $f \in C^1(\mathbb{R})$, $f'(t) = o(|t|^{p-2})$ as $t \to 0$, and $f(0) = 0$;

(f_1) $\displaystyle\lim_{t\to 0} \frac{f(t)}{|t|^{p-1}} = 0$;

(f_2) $\displaystyle\lim_{|t|\to\infty} \frac{|f(t)|}{|t|^{q-1}} = 0, \quad 1 < p \leq q < p^*.$

By using f, problem (D_p) is equivalent to problem

$$\begin{cases} \Delta_p u + \lambda \left[\lambda_\infty \varphi_p(u) + f(u)\right] = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where λ is a bifurcation parameter.

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where λ is a bifurcation parameter. The radial version of this problem is

$$\begin{cases} \left(r^{N-1}\varphi_p(v')\right)' + \lambda\, r^{N-1} \left[\lambda_\infty \varphi_p(v) + f(v)\right] = 0, \\ v'(0) = 0 = v(1), \end{cases} \tag{2}$$

where $v(r) = u(x)$ and $r = ||x||$.

By using *bifurcation theory* we prove

Theorem 1. *Let* $p \geq 2$. *Under hypotheses* $(g_1), (g_2), (g_3)$ *and* (g_4), *problem* (D_p) *has at least* $4j - 1$ *radially symmetric solutions.*

2. Key facts

The following result is a global bifurcation theorem in the radial case.

Theorem 2 (M. del Pino and R. Manasevich). *For each* $k \in \mathbb{N}$ *there is a component* $\mathcal{G}_k \subset \mathbb{R} \times C[0,1]$ *of the set of nontrivial solutions of problem* (2) *whose closure* $\overline{\mathcal{G}_k}$ *contains* $(\mu_k(p)/\lambda_\infty, 0)$. *Moreover,* $\mathcal{G}_k$ *is unbounded in* $\mathbb{R} \times C[0,1]$ *and if* $(\lambda, v) \in \mathcal{G}_k$*, then* v *has exactly* $k-1$ *simple zeroes in (0,1).*

2.1. An uniqueness result for the p- Laplacian

We study the problem

$$\begin{cases} \left(r^{N-1}\varphi_p(u')\right)' + r^{N-1}g(u) = 0, \quad 0 < r < 1 \\ \qquad u(r_0) = \alpha, \quad u'(r_0) = \gamma, \end{cases} \tag{3}$$

for $r_0 \in [0,1)$, where α and γ are real numbers.

u satisfies (3) if and only if u is a fixed point of the operator $T : C[0,1] \longrightarrow C[0,1]$ defined as

$$(Tu)(r) := \alpha + \int_{r_0}^{r} \varphi_{p'} \left[(r_0/t)^{N-1} \varphi_p(\gamma) - \int_{r_0}^{t} (s/t)^{N-1} g(u(s)) ds \right] dt. \tag{4}$$

By using Schauder's fixed point theorem, we proved the following result.

Theorem 3. *Problem* (3) *has a solution* $u \in C^1[0,1]$.

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For $d \in \mathbb{R}$ and $\lambda > 0$ we denote by $u(\cdot, \lambda, d)$ the unique solution of problem

$$\begin{cases} (\varphi_p(u'))' + \dfrac{N-1}{r}\varphi_p\,(u') + \lambda\, g(u) = 0, & 0 < r < 1 \\ u'(0) = 0, \quad u(0) = d. \end{cases} \tag{5}$$

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The continuous dependence of parameters shows that u is a differentiable function in each of the arguments (r, λ, d).

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The continuous dependence of parameters shows that u is a differentiable function in each of the arguments (r, λ, d). If α is a positive number, a simple computation shows that

$$u(r/\alpha, \lambda, d) = u(r, \lambda/\alpha^p, d).$$

Differentiating respect to α and replacing $\alpha = 1$, we find that

$$r\,u'(r,\lambda,d) = p\,\lambda\,u_{\lambda}(r,\lambda,d).$$

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As an application of the Implicit function theorem, the following lemma says that the bifurcation curves are unbounded in the λ-direction.

Lemma 4. *If J is a connected-component of*

$$\{(\lambda, d) : d \neq 0, u(1, \lambda, d) = 0\},$$

then there exist an open interval $(a, b) \subset \mathbb{R} \setminus \{0\}$ and a continuous function $h : (a, b) \to (0, \infty)$ such that $(\lambda, d) \in J$ if and only if $d \in (a, b)$ and $\lambda = h(d)$. In other words J is the graph of h. Moreover, if $a \in \mathbb{R} \setminus \{0\}$ then $\lim_{d \to a} h(d) = \infty$; similarly, if $b \in \mathbb{R} \setminus \{0\}$ then $\lim_{d \to b} h(d) = \infty$.

Because of the uniqueness of solutions for the initial value problem, it is clear that for λ lying in bounded intervals of $(0,\infty)$ problems $(i=1,2)$

$$\begin{cases} \left(r^{N-1}\varphi_p(u')\right)' + \lambda\, r^{N-1}\left[\lambda_\infty \varphi_p(u) + f(u)\right] = 0, \\ \qquad u'(0)=0, \quad u(0)=\beta_i, \end{cases} \tag{6}$$

have a unique solution $u(r)\equiv\beta_i$. Thus, if J is as in Lemma 4 then the domain of h cannot include β_i.

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Also, we prove:

Lemma 5. $(\mu_k(p)/\lambda_\infty, \infty)$ *is a bifurcation point for equation* (2).

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In virtue of a result of García-Melián and Sabina de Lis (actually it is a result of Crandall-Rabinowitz type for p-Laplacian) we can parametrize the branches of solutions which emanate from the trivial solution for every radial eigenvalue $\mu_k(p)/\lambda_\infty$ and also, via involution, the branches from infinity are parametrized.

We denote by $\mathcal{O}_k^+$ the connected component of nontrivial solutions bifurcating from $(\mu_k(p)/\lambda_\infty, 0)$ which contains elements of the form $(\lambda_k(s), s[\phi_k + y_k(s)])$, with $s > 0$. In a similar way we define $\mathcal{O}_k^-$.

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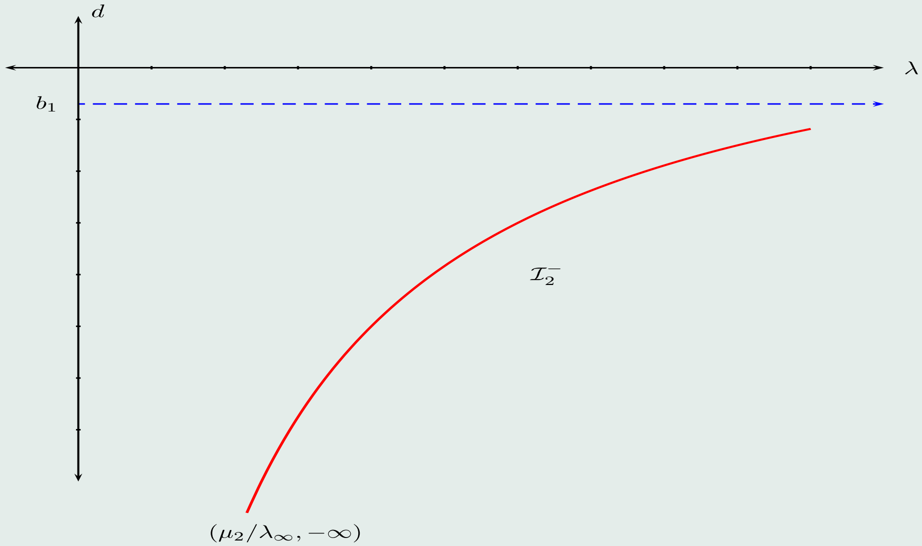
Also, we define $\mathcal{I}_k^+$ as the connected component of nontrivial solutions bifurcating from $(\mu_k(p)/\lambda_\infty, \infty)$ and containing elements of the form $(\lambda_k(s), s[\phi_k + y_k(s)])$, with $s > 0$ large enough. Similarly, we define $\mathcal{I}_k^-$.

Lemma 6. *Let* $J = \{(\lambda, u(0)) : (\lambda, u) \in \mathcal{I}_2^-\}$ *and* h, a, b *be as in Lemma 4. Then* $a = -\infty$ *and* $b < 0$.

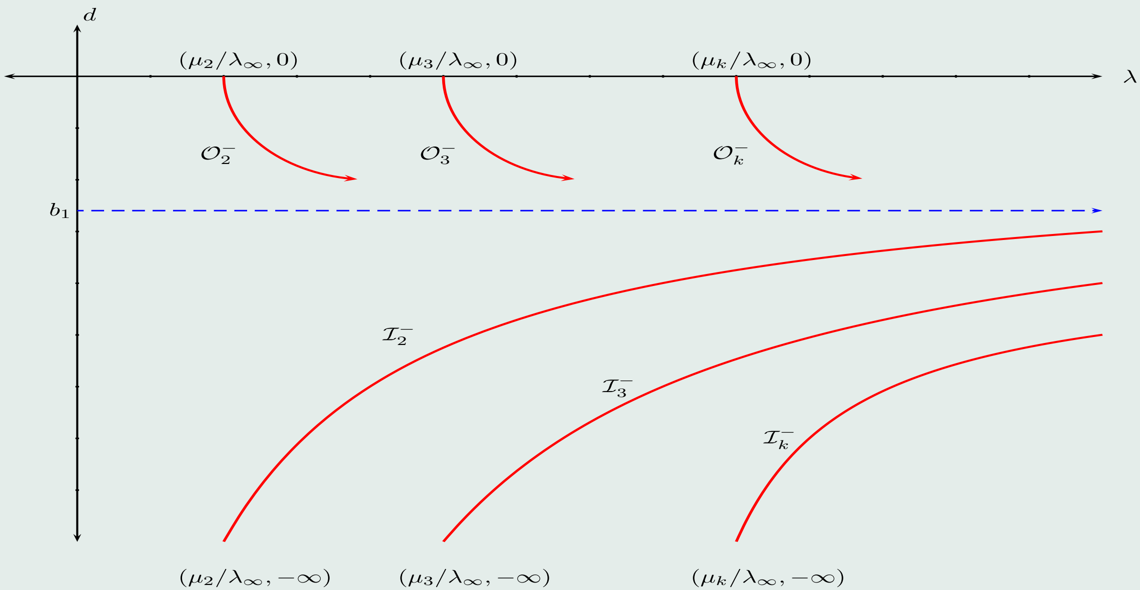
This lemma says that the connected component of nontrivial solutions bifurcating from $(\mu_2(p)/\lambda_\infty, -\infty)$ is bounded from above by a negative constant.


d
λ
b_1
$\mathcal{I}_2^-$
$(\mu_2/\lambda_\infty, -\infty)$

Lemma 7. *If* $J = \{(\lambda, u(0)) : (\lambda, u) \in \mathcal{O}_k^-\}$ *with* $k \geq 2$ *and* a, b *are as in Lemma 4 then* $a \geq b_1$.

Lemma 7. *If* $J = \{(\lambda, u(0)) : (\lambda, u) \in \mathcal{O}_k^-\}$ *with* $k \geq 2$ *and* a, b *are as in Lemma* 4 *then* $a \geq b_1$.

Lemma 8. *If* $J = \{(\lambda, u(0)) : (\lambda, u) \in \mathcal{I}_k^-\}$ *with* $k > 2$ *and* a, b *are as in Lemma* 4 *then* $b \leq b_1$.


d
λ
(μ2/λ∞, 0)
(μ3/λ∞, 0)
(μk/λ∞, 0)
O2−
O3−
Ok−
b1
I2−
I3−
Ik−
(μ2/λ∞, −∞)
(μ3/λ∞, −∞)
(μk/λ∞, −∞)

3. Proof of the theorem 1

Let $k \in \{1, 2, \ldots, j\}$ and $J = \{(\lambda, u(0)) : (\lambda, u) \in \mathcal{O}_k^+\}$ and let (a, b) and h be as in Lemma 4. Due to the definition of $\mathcal{O}_k^+$, we have $a = 0$ and $\lim_{d \to 0} h(d) = \mu_k(p)/\lambda_\infty < 1$.

Because of the uniqueness of the solution of (6), $\mathcal{O}_k^+$ cannot intersect the line $||u|| = \beta$. Consequently $b < \beta$. Lemma 4 implies that $\lim_{d \to b} h(d) = +\infty$ and thus, via mean value theorem, there exists $d_k \in (0, b)$ so that $h(d_k) = 1$.

From this, we have $(1, u(\cdot, 1, d_k)) \in \mathcal{O}_k^+$ is a radial solution of (D_p).

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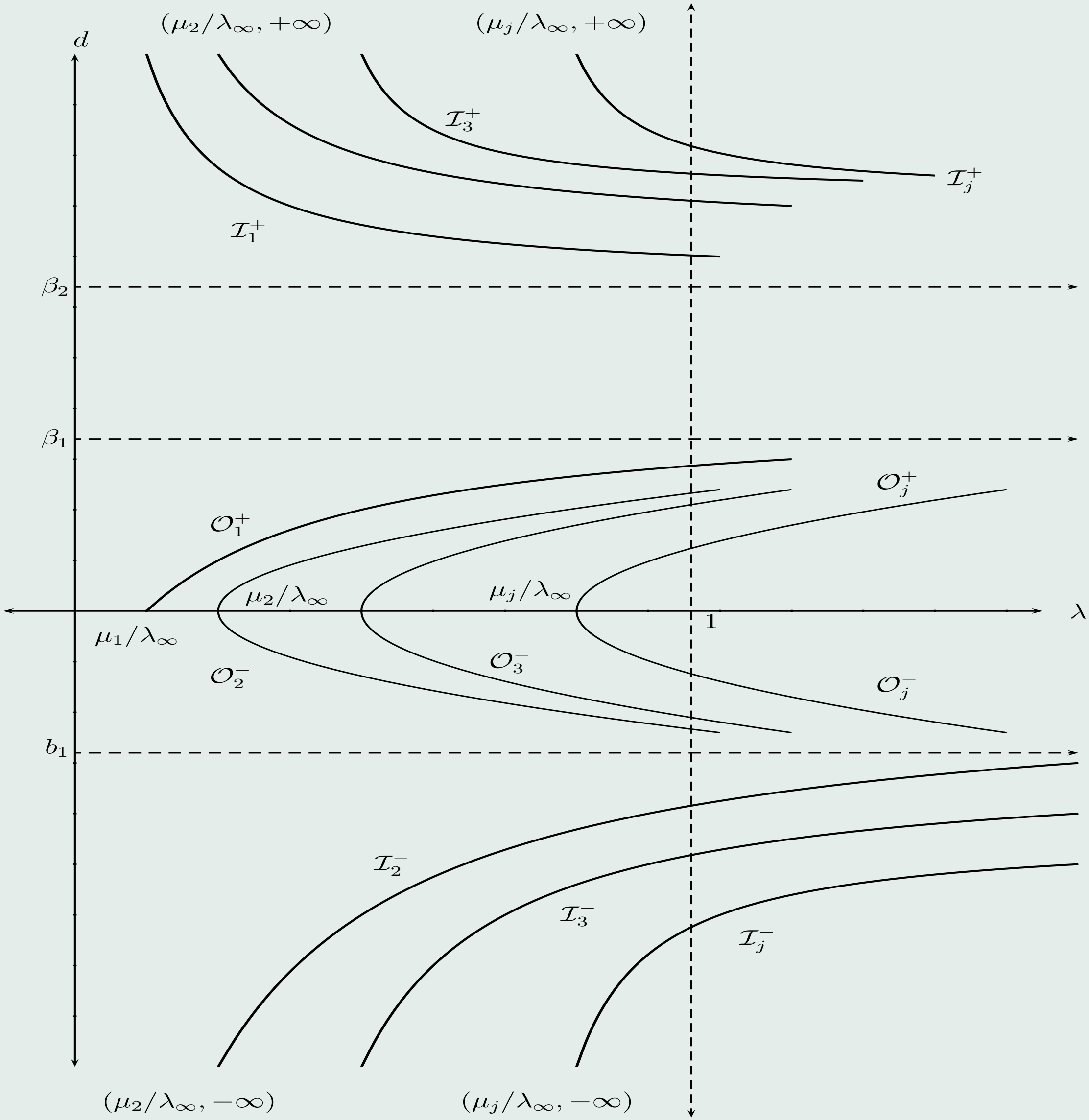
Analogously, if $J = \{(\lambda, u(0)) : (\lambda, u) \in \mathcal{O}_k^-\}$ with $k \in \{2, 3, \ldots, j\}$ and $(a, b), h$ are as in Lemma 4, then $a \geq b_1$ and $\lim_{d \to a} h(d) = +\infty$. As before, there is a $\delta_k \in (a, 0)$ so that $h(\delta_k) = 1$ and hence $(1, u(\cdot, 1, \delta_k)) \in \mathcal{O}_k^-$ is a solution of (2) and so it is a radial solution of (D_p).



$(\mu_2/\lambda_\infty, +\infty)$
$(\mu_j/\lambda_\infty, +\infty)$
d
$\mathcal{I}_3^+$
$\mathcal{I}_j^+$
$\mathcal{I}_1^+$
β_2
β_1
$\mathcal{O}_j^+$
$\mathcal{O}_1^+$
μ_2/λ_∞
μ_j/λ_∞
λ
1
μ_1/λ_∞
$\mathcal{O}_3^-$
$\mathcal{O}_2^-$
$\mathcal{O}_j^-$
b_1
$\mathcal{I}_2^-$
$\mathcal{I}_3^-$
$\mathcal{I}_j^-$
$(\mu_2/\lambda_\infty, -\infty)$
$(\mu_j/\lambda_\infty, -\infty)$

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