The Abstract Wavelet Transform

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Abstract

Given an abstract locally compact topological group, the continuous wavelet transform is defined so that the reconstruction formula is used to prove that the image of the wavelet transform is a reproducing kernel Hilbert space and it is shown also a generalization of the sampling theorem.

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1 Introduction

In one dimension, in order to analyze a signal f on the phase space, we use $\widehat{f}$, the Fourier transform of f. In this case, $\widehat{f}(w) = \int_{\mathbf{R}} e^{-2\pi iwx} f(x)dx$ depends only of the frequency w. That is, $\widehat{f}$ gives information of the frequency w but it does not give the changes of w through the time t. To solve this problem, a function $g(x-t)$ depending of t, called a window is introduced to get the windowed Fourier transform :

$$\big(V_g f\big)(w,t) = \int_{\mathbf{R}} e^{-2\pi iwx}\, f(x)\, g(x-t)\, dx \tag{1}$$

which is a time-frequency localization method. The function g is chosen so that support of g is compact and g is of class C^n for some n. The windowed Fourier transform is then used to determine high and low-frequencies of the signal f in a specific time t.

The continuous wavelet transform of the signal f is also a time-frequency localization method and it is considered as an alternative of the windowed Fourier transform.

For a given a in $\mathbf{R} \setminus \{0\}$, b in $\mathbf{R}$ and a "basic" function h called the mother wavelet,

$$\big(L_h f\big)(a,b) = \int_{\mathbf{R}} f(x)\, \frac{1}{\sqrt{|a|}}\, h\Big(\frac{x-b}{a}\Big)\, dx \qquad (2)$$

is defined as the continuous wavelet transform of f with respect to h.

In this case, a represents the frequency , b the time, and h is taken so that h is in $C_0^\infty(\mathbf{R})$ with $\widehat{h}(0) = 0$.

If we introduce the *translation*, *dilation*, and *modulation* operators defined respectively as : $\big(T_t h\big)(x) = h(x-t)$, $\big(J_a h\big)(x) = \frac{1}{\sqrt{|a|}} h(\frac{x}{a})$, and $\big(E_w h\big)(x) = e^{2\pi i w x} h(x)$, then for f, g, h in $L^2(\mathbf{R})$, the windowed Fourier transform and the continuous wavelet transform, both can be written as inner products :

$$\big(V_g f\big)(w,t) = \big\langle f, \overline{E_{-w} T_t\, g} \big\rangle \text{, and} \quad \big(L_h f\big)(a,b) = \big\langle f, \overline{T_b J_a h} \big\rangle. \quad (3)$$

The difference between these two transforms is with respect to the functions $E_{-w}T_t g$ and $T_b J_a h$. For the windowed Fourier transform, the width of $E_{-w}T_t g$ is the same as for g translated to the proper time of localization and filled it out with high-frequencies. The size of the window $E_{-w}T_tg$ is the same regardless of the value of the frequency w. On the other hand, the functions T_bJ_ah have time width depending on its frequency a. If $a > 1$ and $b = 0$, T_0J_ah detects low-frequencies, and for $0 < a < 1$, T_0J_ah determines high-frequencies. As a consequence, the wavelet transform determines better than the windowed Fourier transform the localization of high and low-frequencies for a specific time t, [1].

2 Notations and definitions

Let us begin by defining a homomorphism for an abelian locally compact topological group. Consider two locally compact topological groups A and B where A is abelian, and consider a homomorphism Γ from B into the group of all automorphisms of A such that the map $(a, b) \to \Gamma_b(a)$ is continuous on $A \times B$ to A. That is, for each b in B, the map $\Gamma_b : A \to A$, $a \to \Gamma_b(a)$ is a homeomorphism.

Definition 1 *Define G as the product of A and B. That is, consider $G = \{(a, b)|a \in A,\, b \in B\}$, and for $(a_1, b_1), (a_2, b_2)$ in G define*

$$(a_1, b_1)(a_2, b_2) = (a_1\Gamma_{b_1}(a_2), b_1b_2) \tag{4}$$

With this product, G becomes a group, where $e = (e_1, e_2)$ is the identity (e_1 is de identity in A and e_2 is the identity in B) and where $(a, b)^{-1} = (\Gamma_{b^{-1}}(a^{-1}), b^{-1})$ is the inverse. Note also that $G = A \times B$ is a locally compact topological group. Then we will denote by $d\mu_G(a, b)$ the left Haar measure on G, the left Haar measure on A by $d\mu_A(a)$ and the left Haar measure in B by $d\mu_B(b)$.

Definition 2 *Given a group G and a set E, an action of G on E is a map $(s, x) \to sx$ of $G \times E \to E$ such that*

1) $ex = x$ *for any* x *in* E *and where* e *is the identity in* G
2) $s(tx) = (st)x$ *for any* x *in* E *and where* s, t *are in* G.

Then we have the following Lemma.

Lemma 1 *The function* $\cdot : G \times A \to A$ *given by* $(a, b) \cdot x = a\Gamma_b(x)$ *is an action of* G *on* A *where* $(a, b) \in G$ *and* $x \in A$.

Definition 3 *Let* G *be a locally compact topological group. The support of the function* $f : G \to \mathbf{C}$ *denoted by supp*(f) *is defined as the closure of* $\{x \in G | f(x) \neq 0\}$, *and* $C_0(G)$ *is defined as the set of continuous functions* $f : G \to \mathbf{C}$ *such that supp*(f) *is compact.*

Definition 4 *For* $1 \leq p < \infty$ *and for a complex valued function defined on the locally compact topological group* A, *define*

$$L^p(A) = \left\{ h : A \to \mathbf{C} \mid \int_A |h(x)|^p d\mu_A(x) < \infty \right\} \tag{5}$$

where $d\mu_A(x)$ *is the left Haar measure on* A.

Then we have the following result for $h \in L^1(A)$ (See [9])

$$\int_A h\left((a, b)^{-1} \cdot x\right) d\mu_A(x) = \chi(a, b) \int_A h(x) d\mu_A(x) \tag{6}$$

Note So, from now on consider $\chi : G \to (0, \infty)$ satisfying (6) for $h \in L^1(A)$.

3 Unitary operators

Definition 5 *For* $h \in L^2(A)$ *define the following operators*

$$(J_a h)(x) = \frac{1}{\sqrt{\chi(a,e_2)}} h((a,e_2)^{-1} \cdot x) \; where \, (a,e_2) \in G, x \in A, a \in A$$

$$(T_b h)(x) = \frac{1}{\sqrt{\chi(e_1,b)}} h((e_1,b)^{-1} \cdot x) \; where \, (e_1,b) \in G, x \in A, b \in B$$

Lemma 2 *For the operators J_a and T_b,*

$$1) \quad J_{a_1} J_{a_2} = J_{a_1 a_2} \quad where \quad a_1, a_2 \in A$$

$$2) \quad T_{b_1} T_{b_2} = T_{b_1 b_2} \quad where \quad b_1, b_2 \in B$$

$$3) \quad T_b J_a = J_{\Gamma_b(a)} T_b \quad and \quad J_a T_b = T_b J_{\Gamma_{b^{-1}}(a)}$$

where $a \in A$ and $b \in B$

Lemma 3 *The operators J_a and T_b are unitary operators*

Proof Let h be in $L^2(A)$. Then by (6),

1)

$$||J_a h||^2 = \int_A |(J_a h)(x)|^2 \, d\mu_A(x) = \int_A (J_a h)(x) \overline{(J_a h)(x)} d\mu_A(x)$$
$$= \int_A \frac{1}{\sqrt{\chi(a,e_2)}} h((a,e_2)^{-1} \cdot x) \frac{1}{\sqrt{\chi(a,e_2)}} \overline{h}((a,e_2)^{-1} \cdot x) d\mu_A(x)$$
$$= \frac{1}{\chi(a,e_2)} \int_A (h\overline{h})((a,e_2)^{-1} \cdot x) d\mu_A(x)$$
$$= \frac{1}{\chi(a,e_2)} \chi(a,e_2) \int_A (h\overline{h})(x) d\mu_A(x) = \int_A |h(x)|^2 d\mu_A(x) = ||h||^2$$

2)

$$\begin{aligned}
||T_b h||^2 &= \int_A |(T_b h)(x)|^2 d\mu_A(x) = \int_A (T_b h)(x)\overline{(T_b h)(x)} d\mu_A(x) \\
&= \int_A \frac{1}{\sqrt{\chi(e_1,b)}} h((e_1,b)^{-1}\cdot x) \frac{1}{\sqrt{\chi(e_1,b)}} \overline{h}((e_1,b)^{-1}\cdot x) \\
&= \frac{1}{\chi(e_1,b)} \int_A (h\overline{h})((e_1,b)^{-1}\cdot x) d\mu_A(x) \\
&= \frac{1}{\chi(e_1,b)} \chi(e_1,b) \int_A (h\overline{h})(x) d\mu_A(x) \\
&= \int_A |h(x)|^2 d\mu_A(x) = ||h||^2
\end{aligned}$$

Moreover, since $J_a^* = J_a^{-1} = J_{a^{-1}}$ and $T_b^* = T_b^{-1} = T_{b^{-1}}$, it follows that both, J_a and T_b are unitary.

This proves Lemma 3.

4 Fourier transform

Definition 6 *Let G be a locally compact topological abelian group, and let $\mathbf{T} = \{z \in \mathbf{C} | |z| = 1\}$. We say that the function $\rho : G \to \mathbf{T}$ is a character on G if ρ is a continuous homomorphism*

Definition 7 *Given a locally compact topological abelian group G, we define the dual group of G as*

$$\hat{G} = \{\rho : G \to \mathbf{T} \,|\, \rho \textit{ is a character}\}$$

In this case we denote $\rho(g) = \langle g, \rho \rangle$ where $g \in G$ and $\rho \in \hat{G}$

Note that $\hat{G}$ is clearly an Abelian group under pointwise multiplication $(\rho_1\rho_2)(g) = \rho_1(g)\rho_2(g)$. Its identity element is the constant function 1 and the inverse element is $\rho^{-1}(g) = \overline{\rho(g)} = \rho(g^{-1})$.

The dual group of a locally compact topological abelian group is used to define an abstract version of the Fourier transform.

Definition 8 *Given $h \in L^1(G)$, the Fourier transform is the function $\hat{h} : \hat{G} \to \mathbf{C}$ defined by*

$$\hat{h}(\rho) = \int_G h(g)\overline{\rho(g)} d\mu_G(g) \tag{7}$$

where the integral is relative to the left Haar measure on G.

Definition 9 *For a function $h \in L^1(\hat{G})$, the inverse Fourier transform of h is the function $\check{h} : G \to \mathbf{C}$ defined as*

$$\check{h}(g) = \int_{\hat{G}} h(\rho)\rho(g) d\mu_{\hat{G}}(\rho) \tag{8}$$

where $d\mu_{\hat{G}}(\rho)$ is the left Haar measure on $\hat{G}$

Note. For $h \in L^1(G)$ and $\hat{h} \in L^1(\hat{G})$ we have

$$h(g) = \int_{\hat{G}} \hat{h}(\rho)\rho(g) d\mu_{\hat{G}}(\rho)$$

Lemma 4 *For $h \in C_0(A)$ we have*

$$\begin{aligned} &1) \quad \widehat{J_a h}(\rho) = \sqrt{\chi(a, e_2)} \quad \overline{\rho(a)} \quad \hat{h}(\rho) \\ &2) \quad \widehat{T_b h}(\rho) = \sqrt{\chi(e_1, b)} \quad \hat{h}(\rho \circ \Gamma_b) \end{aligned}$$

where $\rho \in \hat{A}$, $a \in A$ and $b \in B$.

Proof

1) Note that since

$$\begin{aligned} \widehat{(J_a h)}(\rho) &= \int_A (J_a h)(x)\overline{\rho(x)} d\mu_A(x) \\ &= \int_A \frac{1}{\sqrt{\chi(a, e_2)}} h((a, e_2)^{-1} \cdot x)\overline{\rho(x)} d\mu_A(x) \end{aligned}$$

and

$$\begin{aligned}\rho(x) &= \rho(aa^{-1}\Gamma_{e_2}(x)) = \rho(a)\rho(a^{-1}\Gamma_{e_2}(x)) \\ &= \rho(a)\rho((a^{-1}, e_2)\cdot x) = \rho(a)\rho((a, e_2)^{-1}\cdot x)\end{aligned}$$

it follows from (6) that

$$\begin{aligned}\widehat{J_a h}(\rho) &= \int_A \frac{1}{\sqrt{\chi(a,e_2)}} h((a,e_2)^{-1}\cdot x)\overline{\rho(a)\rho((a,e_2)^{-1}\cdot x)}d\mu_A(x) \\ &= \frac{1}{\sqrt{\chi(a,e_2)}}\overline{\rho(a)}\int_A (h\overline{\rho})((a,e_2)^{-1}\cdot x)d\mu_A(x) \\ &= \frac{1}{\sqrt{\chi(a,e_2)}}\overline{\rho(a)}\chi(a,e_2)\int_A (h\overline{\rho})(x)d\mu_A(x) \\ &= \sqrt{\chi(a,e_2)} \quad \overline{\rho(a)}\int_A h(x)\overline{\rho(x)}d\mu_A(x) \\ &= \sqrt{\chi(a,e_2)} \quad \overline{\rho(a)} \quad \hat{h}(\rho)\end{aligned}$$

2) Similarly, since

$$\begin{aligned}\widehat{(T_b h)}(\rho) &= \int_A (T_b h)(x)\overline{\rho(x)}d\mu_A(x) = \\ &= \int_A \frac{1}{\sqrt{\chi(e_1,b)}} h((e_1,b)^{-1}\cdot x)\overline{\rho(x)}d\mu_A(x)\end{aligned}$$

and

$$\begin{aligned}\rho(x) &= \rho\left[\Gamma_{bb^{-1}}(x)\right] = \rho\left[\Gamma_b(\Gamma_{b^{-1}}(x))\right] = \rho\left[\Gamma_b(e_1\Gamma_{b^{-1}}(x))\right] \\ &= \rho\left[\Gamma_b((e_1,b^{-1})\cdot x)\right] = \rho\left[\Gamma_b((e_1,b)^{-1}\cdot x)\right] \\ &= (\rho\circ\Gamma_b)((e_1,b)^{-1}\cdot x)\end{aligned}$$

It follows from (6) that

$$\begin{aligned}
\widehat{T_b h}(\rho) &= \int_A \frac{1}{\sqrt{\chi(e_1,b)}} h((e_1,b)^{-1}\cdot x)\overline{(\rho\circ\Gamma_b)((e_1,b)^{-1}\cdot x)}d\mu_A(x) \\
&= \frac{1}{\sqrt{\chi(e_1,b)}} \int_A (h\cdot \overline{\rho\circ\Gamma_b})((e_1,b)^{-1}\cdot x)d\mu_A(x) \\
&= \frac{1}{\sqrt{\chi(e_1,b)}}\chi(e_1,b) \int_A (h\cdot \overline{\rho\circ\Gamma_b})(x)d\mu_A(x) \\
&= \sqrt{\chi(e_1,b)} \quad \int_A h(x)\overline{(\rho\circ\Gamma_b)(x)}d\mu_A(x) \\
&= \sqrt{\chi(e_1,b)} \qquad \hat{h}(\rho\circ\Gamma_b)
\end{aligned}$$

This proves Lemma 4.

Corollary 1 *Let h be in $L^1(A)$. Then*

$$\widehat{J_a T_b h}(\rho) = \sqrt{\chi(a,b)} \quad \overline{\rho(a)} \quad \hat{h}(\rho\circ\Gamma_b)$$

Corollary 2 *Let h be in $L^1(A)$. Then*

$$\begin{aligned}
&1) \quad \widehat{J_a h}(\rho) = J_{a^{-1}}\hat{h}(\rho) \\
&2) \quad \widehat{T_b h}(\rho) = T_{b^{-1}}\hat{h}(\rho) \\
&3) \quad \widehat{J_a T_b h}(\rho) = J_{a^{-1}}T_{b^{-1}}\hat{h}(\rho)
\end{aligned}$$

5 Unitary representation

Definition 10 *For (a,b) in $G = A\times B$, define the two - parameter family of operators $U(a,b) = J_a T_b$. Note that $U(a,b)$ acts on the*

Hilbert space $L^2(A)$ *by:*

$$
\begin{aligned}
&(U(a,b)h)(x) = (J_a T_b h)(x) = (J_a(T_b h))(x)\\
&= \frac{1}{\sqrt{\chi(a,e_2)}}(T_b h)((a,e_2)^{-1}\cdot x)\\
&= \frac{1}{\sqrt{\chi(a,e_2)}}\frac{1}{\sqrt{\chi(e_1,b)}}h((e_1,b)^{-1}(a,e_2)^{-1}\cdot x)\\
&= \frac{1}{\sqrt{\chi((a,e_2)(e_1,b))}}h\left[((a,e_2)(e_1,b))^{-1}\cdot x\right]\\
&= \frac{1}{\sqrt{\chi(a,b)}}h((a,b)^{-1}\cdot x)
\end{aligned}
$$

Lemma 5 . *$U(a,b) = J_a T_b$ is a unitary representation of G acting on the Hilbert space $L^2(A)$.*

Proof

Note that since the operators J_a and T_b are unitary operators (Lemma 3), it follows that $U(a,b)$ is unitary.

Now, let us prove that $U(a,b)$ is a representation of G acting on $L^2(A)$.

On one hand, from Lemma 2,

$$
\begin{aligned}
&U((a_1,b_1)(a_2,b_2)) = U(a_1\Gamma_{b_1}(a_2), b_1 b_2) = J_{a_1\Gamma_{b_1}(a_2)}T_{b_1 b_2}\\
&= J_{a_1}J_{\Gamma_{b_1}(a_2)}T_{b_1}T_{b_2} = J_{a_1}T_{b_1}J_{a_2}T_{b_2} = U(a_1,b_1)U(a_2,b_2)
\end{aligned}
$$

On the other hand, since $U(e_1,e_2) = J_{e_1}T_{e_2} = I$, where I is the identity operator, it follows that $U(a,b)$ is a representation of G.

This proves Lemma 5

Lemma 6 *The left Haar measure on $G = A \times B$ is*

$$d(a,b) = \frac{1}{\chi(a,b)}d\mu_A(a)d\mu_B(b)$$

Proof

Let h be in $L^1(G)$, then by (6)

$$\int_G h\left[(a_0,b_0)^{-1}(a,b)\right]d\mu_A(a)d\mu_B(b) = \chi(a_0,b_0)\int_G h(a,b)d\mu_A(a)d\mu_B(b)$$

Now, replacing h by $\frac{h}{\chi}$ we have

$$\int_G \frac{h\left[(a_0,b_0)^{-1}(a,b)\right]}{\chi\left[(a_0,b_0)^{-1}(a,b)\right]}d\mu_A(a)d\mu_B(b) = \chi(a_0,b_0)\int_G \frac{h(a,b)}{\chi(a,b)}d\mu_A(a)d\mu_B(b)$$

Then

$$\int_G h\left[(a_0,b_0)^{-1}(a,b)\right]\frac{1}{\chi(a,b)}d\mu_A(a)d\mu_B(b) = \int_G h(a,b)\frac{1}{\chi(a,b)}d\mu_A(a)d\mu_B(b)$$

That is

$$\int_G h\left[(a_0,b_0)^{-1}(a,b)\right]d(a,b) = \int_G h(a,b)d(a,b)$$

This shows that $d(a,b)$ is a left Haar measure on G.

This proves Lemma 6.

6 Admissibility condition

Definition 11 *A function h in $L^2(A)$ is said to be admissible if*

$$\int_G |\langle h, U(a,b)h\rangle|^2 d(a,b) < \infty$$

Lemma 7 *Let h be in $L^1(A)\cap L^2(A)$. If $\mu(B)<\infty$, then*

$$C_h \equiv \int_B |\hat{h}(\rho\circ\Gamma_b)|^2 d\mu_B(b)$$

is uniformly bounded for $\rho \in \hat{A}$

Proof Note that since

$$\hat{h}(\rho \circ \Gamma_b) = \int_A h(x)\overline{(\rho \circ \Gamma_b)(x)}d\mu_A(x),$$

it follows that

$$|\hat{h}(\rho \circ \Gamma_b)| \leq \int_A |h(x)|\,|\rho(\Gamma_b(x))|d\mu_A(x) = ||h||_1$$

Hence, $C_h \leq ||h||_1^2\,\mu(B) < \infty$. This proves Lemma 7.

Lemma 8 *Let h be in $L^1(A) \cap L^2(A)$ if*

$$0 < C_h \equiv \int_B |\hat{h}(\rho \circ \Gamma_b)|^2 d\mu_B(b)$$

is uniformly bounded for $\rho \in \hat{A}$, then h is admissible.

7 The continuous wavelet transform

Definition 12 *Given (a, b) in $G = A \times B$ and h admissible in $L^2(A)$, the continuous wavelet transform with respect to h is defined as the linear operator*

$$L_h(a, b) : L^2(A, d\mu_A) \to L^2(G, d(a, b))$$

such that for any f in $L^2(A)$ we have

$$(L_h f)(a, b) = \langle f, U(a, b)h\rangle_{L^2(A)}$$

That is,

$$(L_h f)(a, b) = \int_A f(x)\overline{[U(a, b)h]\,(x)}d\mu_A(x)$$

$$= \int_A f(x)\frac{1}{\sqrt{\chi(a, b)}}\overline{h((a, b)^{-1} \cdot x)}d\mu_A(x)$$

Note In order to get back the function f from the continuous wavelet transform $(L_h f)(a,b)$, we will apply the Grossmann-Morlet-Paul theorem [2], where the hypotheses for the representation $U(a,b)$ are: unitary, irreducible and strongly continuous. In our case, our representation is unitary, so the following lemmas will show that $U(a,b) = J_a T_b$ is irreducible and strongly continuous.

Lemma 9 *The representation $U(a,b)$ of the group $G = A \times B$ acting on $L^2(A)$ is irreducible.*

Proof Suppose $h \in L^2(A) \setminus \{0\}$ and suppose $f \in L^2(A)$ is such that $\langle f, U(a,b)h \rangle = 0$ for all (a,b) in G. To show the representation $U(a,b)$ is irreducible we will show that $f = 0$ in $L^2(A)$.

Under the assumptions we have

$$\int_G |\langle f, U(a,b)h \rangle|^2 d(a,b) = 0,$$

but ,

$$0 = \int_G |\langle f, U(a,b)h \rangle|^2 d(a,b) = C_h ||f||^2$$

Since h is not identically zero, it follows that $||f|| = 0$. Thus $f = 0$

This proves Lemma 9.

Definition 13 *For a function $h : A \to \mathbf{C}$, define the left and right translations of h by*

$$(I_a h)(x) = h(a^{-1}x) \quad \textit{and} \quad (D_a h)(x) = h(xa) \quad \textit{where} \quad a, x \in A$$

Definition 14 *For a function $h : A \to \mathbf{C}$, we say that:*

a) h is left uniformly continuous if $||I_a h - h||_\infty \to 0$ as $a \to e_1$

b) h is right uniformly continuous if $||D_a h - h||_\infty \to 0$ as $a \to e_1$, where $||\quad||_\infty$ is the uniform norm

Lemma 10 *(See [3]) If $h \in C_0(A)$, then h is left and right uniformly continuous*

Lemma 11 *If h is in $C_0(A)$, then*
a) $||J_a h - h||_\infty \to 0$ *as* $a \to e_1$
b) $||T_b h - h||_\infty \to 0$ *as* $b \to e_2$

Lemma 12 *Let h be in $L^2(A)$. Then*

1) $||J_a h - h||_2 \to 0$ *as* $a \to e_1$

2) $||T_b h - h||_2 \to 0$ *as* $b \to e_2$

Lemma 13 *Let h be in $L^2(A)$. Then $||U(a,b)h - h||_2 \to 0$ as $(a,b) \to (e_1, e_2)$*

Proof By Lemma 12,

$$||U(a,b)h - h||_2 = ||J_a T_b h - h||_2 = ||J_a T_b h - J_a h + J_a h - h||_2$$
$$\leq ||J_a(T_b h - h)||_2 + ||J_a h - h||_2 = ||T_b h - h||_2 + ||J_a h - h||_2 \to 0$$

as $a \to e_1$ and $b \to e_2$.

This proves Lemma 13.

Lemma 14 *The representation $U(a,b)$ is strongly continuous.*

Proof Le us prove that $||U(a,b)h - U(a_1,b_1)h||_2 \to 0$ as $(a,b) \to (a_1,b_1)$ for any h in $L^2(A)$.

Consider a, a_1 in A. Then for b, b_1 in B,

$$J_a T_b h = J_a T_b \left(T_{b_1^{-1}} (J_{a_1^{-1}} J_{a_1}) T_{b_1} \right) h$$
$$= (J_a T_b T_{b_1^{-1}} J_{a_1^{-1}})(J_{a_1} T_{b_1} h) = (J_a T_{bb_1^{-1}} J_{a_1^{-1}})(J_{a_1} T_{b_1} h)$$
$$= J_a (J_{\Gamma_{bb_1^{-1}}(a_1^{-1})} T_{bb_1^{-1}})(J_{a_1} T_{b_1} h) = (J_{a\Gamma_{bb_1^{-1}}(a_1^{-1})} T_{bb_1^{-1}})(J_{a_1} T_{b_1} h)$$

Then for $u = J_{a_1}T_{b_1}h$,

$$\begin{aligned} J_aT_bh - J_{a_1}T_{b_1}h &= J_{a\Gamma_{bb_1^{-1}}(a_1^{-1})}T_{bb_1^{-1}}u - u \\ &= J_{a\Gamma_{bb_1^{-1}}(a_1^{-1})}T_{bb_1^{-1}}u - J_{a\Gamma_{bb_1^{-1}}(a_1^{-1})}u + J_{a\Gamma_{bb_1^{-1}}(a_1^{-1})}u - u \\ &= J_{a\Gamma_{bb_1^{-1}}(a_1^{-1})}\left[T_{bb_1^{-1}}u - u\right] + \left[J_{a\Gamma_{bb_1^{-1}}(a_1^{-1})}u - u\right] \end{aligned}$$

Then by Lemma 12,

$$\begin{aligned} &||J_aT_bh - J_{a_1}T_{b_1}h||_2 \\ &\leq ||J_{a\Gamma_{bb_1^{-1}}(a_1^{-1})}\left[T_{bb_1^{-1}}u - u\right]||_2 + ||J_{a\Gamma_{bb_1^{-1}}(a_1^{-1})}u - u||_2 \\ &= ||T_{bb_1^{-1}}u - u||_2 + ||J_{a\Gamma_{bb_1^{-1}}(a_1^{-1})}u - u||_2 \to 0 \quad \text{as} \quad (aa_1^{-1}, bb_1^{-1}) \to (e_1, e_2) \end{aligned}$$

Thus, $||J_aT_bh - J_{a_1}T_{b_1}h||_2 \to 0$ as $(a, b) \to (a_1, b_1)$.

This proves Lemma 14.

8 Reconstruction formula

Lemma 15 . *For any f, g in $L^2(A)$ and an admissible non-zero function h in $L^2(A)$, we have the following identity in the weak sense*

$$f = \frac{1}{C_h}\int_G (L_hf)(a, b)\, U(a, b)h\, d(a, b) \tag{9}$$

Proof The representation $U(a, b)$ is a strongly continuous irreducible unitary representation of the locally compact topological group $G = A \times B$ acting on the Hilbert space $L^2(A)$. So, if there is a non-zero admissible vector h in $L^2(A)$, then by the Grossmann-Morlet-Paul theorem [2], for f, g in $L^2(A)$,

$$\int_G \langle f, U(a, b)h\rangle\, \overline{\langle g, U(a, b)h\rangle} d(a, b) = C_h \langle f, g\rangle \tag{10}$$

Hence,

$$f = \frac{1}{C_h} \int_G (L_h f)(a,b)\, U(a,b) h \, d(a,b)$$

in the weak sense.

This proves Lemma 15.

9 Plancherel's Theorem

Lemma 16 *Parseval's formula. For f, g in $L^2(A)$,*

$$\langle f, g \rangle_{L^2(A)} = \frac{1}{C_h} \langle L_h f, L_h g \rangle_{L^2(G)} \tag{11}$$

Proof It comes from (10)

Lemma 17 *Palncherel's Theorem. For f, g in $L^2(A)$,*

$$||f||^2_{L^2(A)} = \frac{1}{C_h} ||L_h f||^2_{L^2(G)} \tag{12}$$

Proof If $f = g$, then from Lemma 16

$$\langle f, f \rangle_{L^2(A)} = \frac{1}{C_h} \langle L_h f, L_h f \rangle_{L^2(G)}$$

That is,

$$||f||^2_{L^2(A)} = \frac{1}{C_h} ||L_h f||^2_{L^2(G)}$$

which means

$$\int_A |f(a)|^2 d\mu_A(a) = \frac{1}{C_h} \int_G |(L_h f)(a,b)|^2 d(a,b)$$

This proves Lemma 17

10 Reproducing kernel Hilbert space

We now give a characterization of the image of the wavelet transform by a reproducing kernel. Note that not every function $F(a,b)$ in $L^2(G)$ is the wavelet transform of some function f in $L^2(A)$. That is,

$$Im(L_h(a,b)) = \{F(a,b) | (L_h f)(a,b) = F(a,b) \text{ for some } f \in L^2(A)\}$$

is a proper subspace of $L^2(G)$.

To see this, note that for $F(a,b) = (L_h f)(a,b)$, we have $F(a,b)$ is bounded since

$$|F(a,b)| = |(L_h f)(a,b)| = |\langle f, U(a,b)h\rangle| \leq ||f||_2 ||h||_2$$

Hence, any unbounded and square integrable function $F(a,b)$ is not in $Im(L_h(a,b))$. Then we have the following Lemma.

Lemma 18 *The image of the wavelet transform with respect to an admissible function h in $L^2(A)$ is the closed subspace of functions $F(a,b)$ in $L^2(G)$ that satisfy*

$$F(a,b) = \frac{1}{C_h} \int_G F(a_0,b_0) K(a,b;a_0,b_0) d(a_0,b_0)$$

where

$$K(a,b;a_0,b_0) = \overline{(L_h U(a,b)h)(a_0,b_0)}$$

is the reproducing kernel associated with h.

Proof If F is in $Im(L_h(a,b))$, there is $f \in L^2(A)$ such that $(L_h f)(a,b) = F(a,b)$. Then by (11) with $g = U(a,b)h$,

$$F(a,b) = (L_h f)(a,b) = \langle f, U(a,b)h\rangle = \frac{1}{C_h} \langle L_h f, L_h g\rangle$$

$$= \frac{1}{C_h} \int_G (L_h f)(a_0,b_0) \overline{(L_h g)(a_0,b_0)} d(a_0,b_0)$$

Now by taking

$$K(a,b;a_0,b_0) = \overline{(L_h g)(a_0,b_0)}$$

we have

$$F(a,b) = \frac{1}{C_h}\int_G F(a_0,b_0)K(a,b;a_0,b_0)d(a_0,b_0) \qquad (13)$$

This shows that the image of $L_h(a,b)$ is a reproducing kernel Hilbert space embedded as a close subspace of $L^2(G, \frac{1}{C_h}d(a,b))$, where

$$K(a,b;a_0,b_0) = \overline{[L_h J_a T_b h]\,(a_0,b_0)} = \overline{(L_h h)(\Gamma_{b^{-1}}(a^{-1}a_0), b^{-1}b_0)}$$

is the reproducing kernel

This proves Lemma 18.

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