Growth!

A basic, continuous model of population dynamics describes how its size changes in time. For example:

$$\frac{dg}{dt} = f(g)$$

where

- g represents the population.
- t represents time.
- f(g) is the "growth function".

We will review two well known examples: exponential and logistic growth. The **exponential growth** model looks like:

$$\frac{dp}{dt} = ap.$$

In this equation,

- p represents the population.
- t represents time.
- a is the growth (or decay) rate of the population.

The exponential growth equation has the general solution:

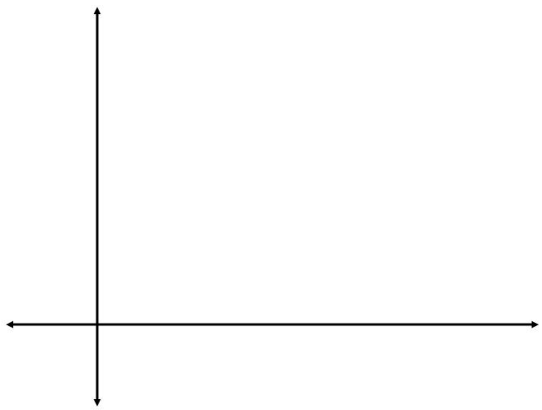
$$p(t) = p(0)e^{at}.$$

(For review, see *Schaum's Outline, Differential Equations.*) This solution can also be written as

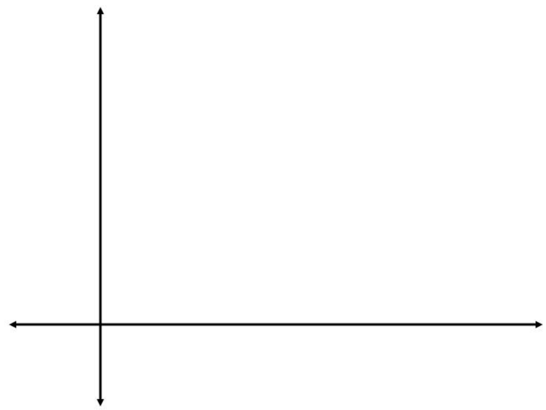
$$p(t_0) = p(0)e^{a(t-t_0)}.$$

(*Can you verify this?*) On the following graphs, we want to sketch (a) $\frac{dp}{dt}$ as a function of p, and (b) the general solution, $p(t)$ as a function of t:

(a)



(b)



The **logistic growth** model for a population looks like:

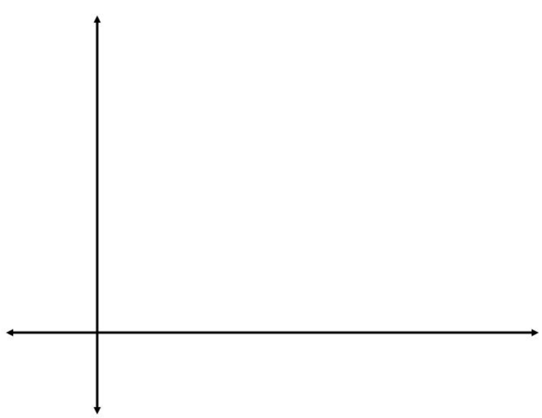
$$\frac{dq}{dt} = bq(1 - \frac{q}{k}).$$

In this case,

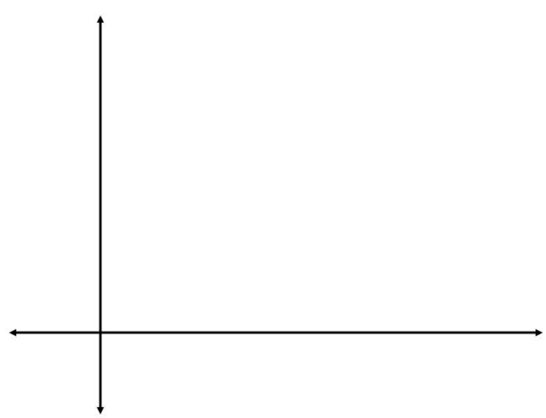
- q represents the population.
- t represents time.
- b is the intrinsic growth rate of the population.
- k is the *carrying capacity* of the population.

On the following graphs, we want to sketch (a) $\frac{dq}{dt}$ as a function of q, and (b) the general solution, $q(t)$ as a function of t *without explicitly finding the general solution!* You can try, for example, $k = 5$.

(a)



(b)



How does the logistic behavior differ from the exponential growth behavior? What does this mean in terms of the population change over time?

For You!

Exercise 1. [1] It has been observed both *in vivo* and *in vitro* that solid tumors experience an initial period of quick growth, followed by a period when growth slows or stops, followed by another period of growth. It has been suggested that the first period of growth is during the *avascular phase*, when the tumor cells must acquire nutrient through diffusion from outside the tumor. Once the tumor reaches a certain size, the tumor cells release *angiogenic growth factors* which stimulate the growth of blood vessels towards the tumor which eventually reach the interior of the tumor. After this vascularization, the tumor undergoes another period of growth. In this exercise, the angiogenic process will be modeled by a drastic **slowing** in the growth rate when a tumor reaches a certain size, T_α.

(a) Sketch a graph of a possible growth function $F(t) = \frac{dT}{dt}$ which is positive for $0 < T < T_{max}$ and is very small for T near $T_\alpha < T_{max}$. In the spirit of generating tractable models, you may want to make your function as simple as possible, within the prescribed constraints.

(b) Write a differential equation for $T(t)$, after writing the function for $F(t)$ that you found in part (a).

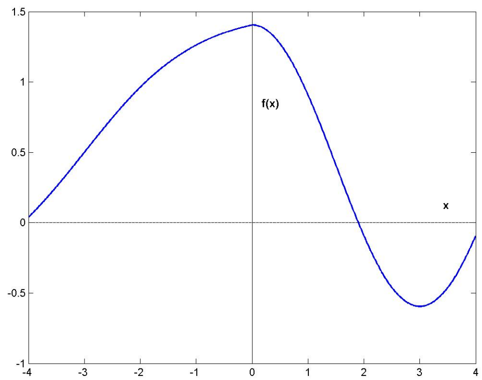
Exercise 2. [1] It has been observed that in certain tumors grown *in vitro*, only a thin layer of cells on the tumor's surface are actually proliferating. Consider a perfectly spherical tumor, and let $T(t)$ denote the tumor population at time t.

(a) Write a differential equation for $T(t)$, assuming that only the cells on the surface of the sphere proliferate. (Assume that the number of cells, T, is proportional to volume, but that the number of proliferating cells is proportional to the surface area of the sphere. You will need to express the number of proliferating cells as a function of T.)

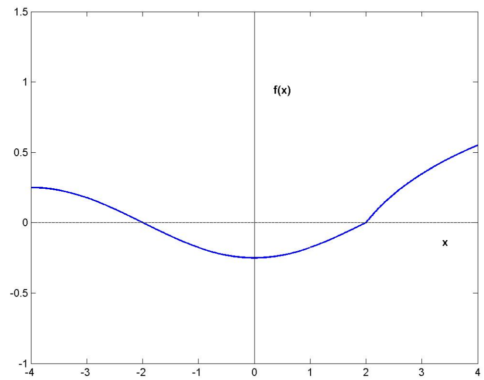
[1] de Pillis and Radunskaya, 2004

Exercise 3. [2] Suppose the function $x(t)$ evolves according to the following differential equation: $\frac{dx}{dt} = f(x)$. For each of the following candidate functions $f(x)$, describe what happens to $x(t)$ as t gets large if (a) $x(0) = 1$; (b) $x(0) = -4$. What about other values of $x(0)$?

(A)



(B)



References

1. de Pillis, L.G. and Radunskaya, A.E. *Exercises for Tumor Dynamics Module*. 2004.
2. Taubes, C. *Modeling differential equations in biology.* Prentice Hall, Inc., 2001.

[2] Taubes, 2001