Systems, Flows and Stability

We are going to consider

$$\dot{x} = f(x), \tag{1}$$

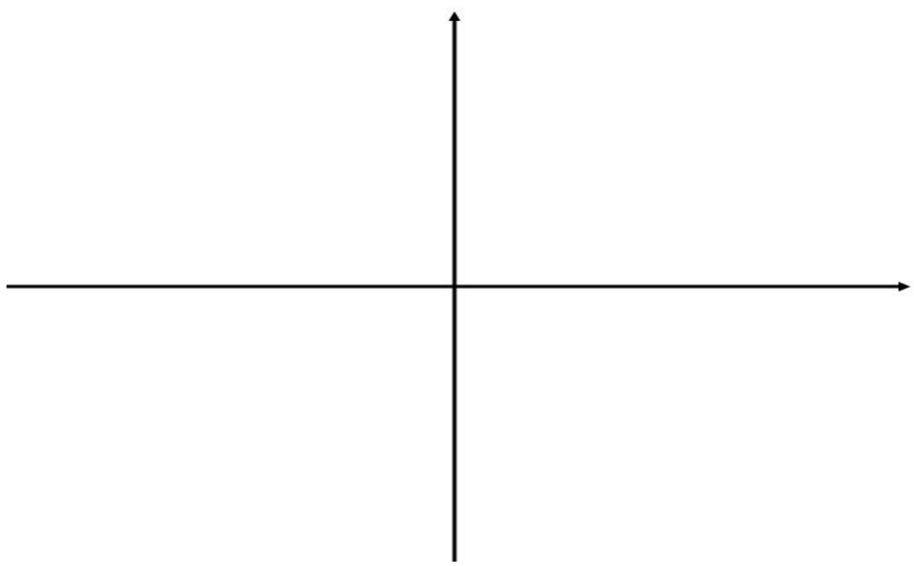
where x is a continuous function of t and $f(x)$ is a *smooth* function of x and does not depend explicitly on t. In Strogatz terminology, we can refer to equation 1 as a **first-order system**–this is in the sense of a dynamical system and not a system of equations.

More specifically, let's consider $\dot{x} = \sin(x)$. This differntial equation has the implicit solution:

$$t = \ln\left|\frac{\csc(x_0) + \cot(x_0)}{\csc(x) + \cot(x)}\right|.$$

How helpful is this solution in terms of explaining what happens to x as t changes?

We use a different approach. Let's start by sketching $\dot{x}$ as a function of x on the following graph:



For physical intuition, let's consider x as the position of a particle on the real line. Then,

What is $\dot{x}$? The velocity vector at each position x–it represents a **vector field** on the line.

When does the flow move to the right? to the left? It moves to the right when $\sin(x) > 0$ and to the left when $\sin(x) < 0$.

What happens at points where $x = n\pi$? There is no flow here, these are **fixed points**.

About those fixed points:

Which ones are stable? Fixed points where the flow comes into the point on both sides. These are also called *attractors, sinks*.

Which ones are unstable? Fixed points where the flow is away from the point on both sides. These are also called *repellers, sources*.

What might be a half-stable or hybrid case? Fixed points where flow comes in on one side and leaves on another are hybrids and are often referred to as half-stable.

Using the information above, sketch some trajectories of x versus t on the following graph:

For You!

Exercise 1.[1] Analyze the following equations graphically. In each case, sketch the vector field on the real line, find all the fixed points, classify their stability, and sketch the graph of $x(t)$ for different initial conditions.

(a) $\dot{x} = x - x^3$

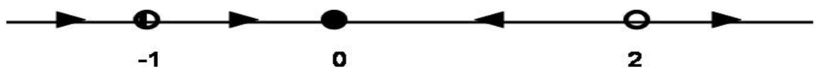
(b) $\dot{x} = x^2(6 - x)$

(c) $\dot{x} = 0$

Exercise 2.[1] (Working backwards, from flows to equations) Given an equation $f(x)$, we know how to sketch the corresponding flow on the real line. Here you are asked to solve the opposite problem: for the following phase portrait, find an equation that is

[1]Strogatz, 1994

consistent with it. (There are an infinite number of correct answers–and wrong ones too!)



Exercise 3. [1] The growth of cancerous tumors can be modeled by the Gompertz law $\dot{N} = -aN\ln(bN)$, where $N(t)$ is proportional to the number of cells in the tumor and $a, b > 0$ are parameters.

(a) Interpret a and b biologically.

(b) Sketch the vector field and then graph $N(t)$ for various initial values. The predictions of this simple model agree surprisingly well with data on tumor growth, as long as N is not too small; see Aroesty et al. (1973) and Newton (1980) for examples.

Exercise 4.[2] Another simple model of tumor/host interaction describes the growth of two populations, each growing according to a logistic law and competing with each other for resources. In this model, we lump together all non-tumor cells which are at the tumor site, including normal tissue as well as immune cells. We do *not* assume a constant source of immune cells.

Let $X(t)$ denote the normal cell population at time t, including immune cells, and let $Y(t)$ denote the tumor population at time t. The system of differential equations which describes the model is:

$$\begin{aligned} \frac{dX}{dt} &= a_1X(1-b_1X) - c_1XY \\ \frac{dY}{dt} &= a_2Y(1-b_2Y) - c_2XY \end{aligned}$$

(a) What is the biological interpretation of each of the parameters a_1, a_2, b_1, b_2, c_1 and c_2? Are they necessarily all positive or negative?

[2] de Pillis and Radunskaya, 2004

(b) Describe hypothetical experiments which would allow the determination of these parameters.

References

1. de Pillis, L.G. and Radunskaya, A.E. *Exercises for Tumor Dynamics Module.* 2004.
2. Strogatz, S. *Nonlinear Dynamics and Chaos.* Perseus Books Publishing, LLC., 1994.