Phase Planes and MATLAB^{TM}!

Given a system of autonomous[1] differential equations, it is often useful to sketch the 2D vector field or phase plane. MATLAB^{TM} is a useful tool to create phase planes. So today we will play around with phase planes in MATLAB^{TM}.

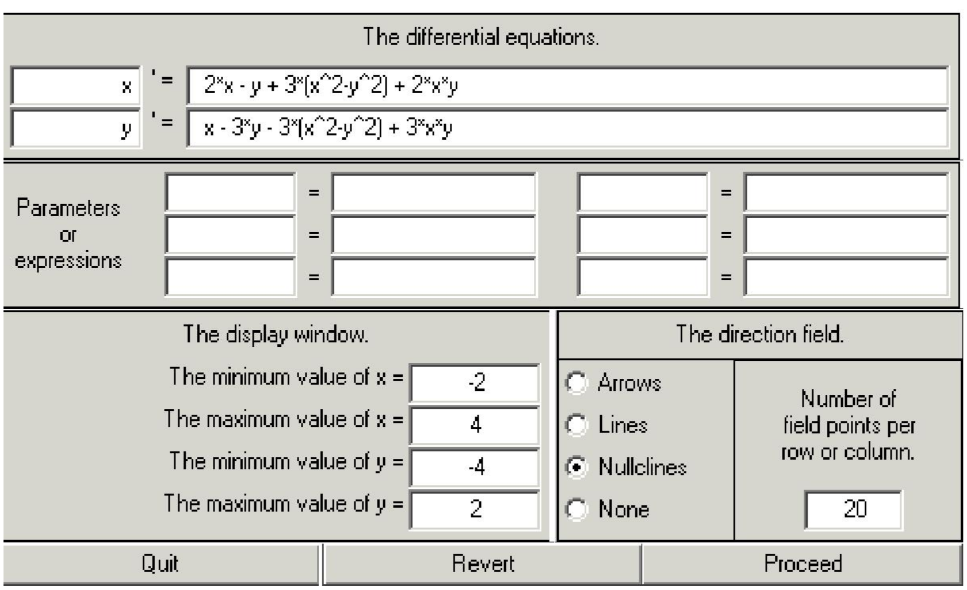
Fortunately, John Polking from Rice University[2] has written a MATLAB^{TM} code that we will use to investigate phase planes. To begin, click on the MATLAB^{TM} icon in order to open a MATLAB^{TM} command window. Change the directory to the public server (directory P:) in the cancer_matlab directory (the working directory line should read P:cancer_matlab). At the prompt, type:

pplane7

and hit return. A window entitled "PPLANE6 Setup" should open. Note that it already has values entered for the x' and y' values indicating that we are working with the system:

$$\begin{aligned}\frac{dx}{dt} &= 2x - y + 3(x^2 - y^2) + 2xy,\\ \frac{dy}{dt} &= x - 3y - 3(x^2 - y^2) + 3xy.\end{aligned}$$

Notice that the range of x and y values at which to plot arrows is already set for $-2 \le x \le 4$ and $-4 \le y \le 2$. If we want to consider the behavior over a different range of values, we will need to change this. But for now, it is fine. First click on the circle indicating "nullclines", the window should look like the following:



[1]Recall:**autonomous** means the differential equations do not depend explicitly on time.

[2]Polking, web

Then, click "proceed". You should have a figure pop up with pink and orange nullclines. The arrows on the nullclines indicate the direction of motion there. Using these arrows, can you determine which is the x-nullcline and which is the y-nullcline?

Return to the PPLANE6 Setup window. Now click the circle to choose arrows. The PPLANE6 Display window should now show the vector field for this system. The arrows indicate the direction of motion at the point of their tail. The arrows are also scaled to indicate their relative magnitudes. (If you had selected the circle to choose lines, you would see the same field but without direction and scale indicated.)

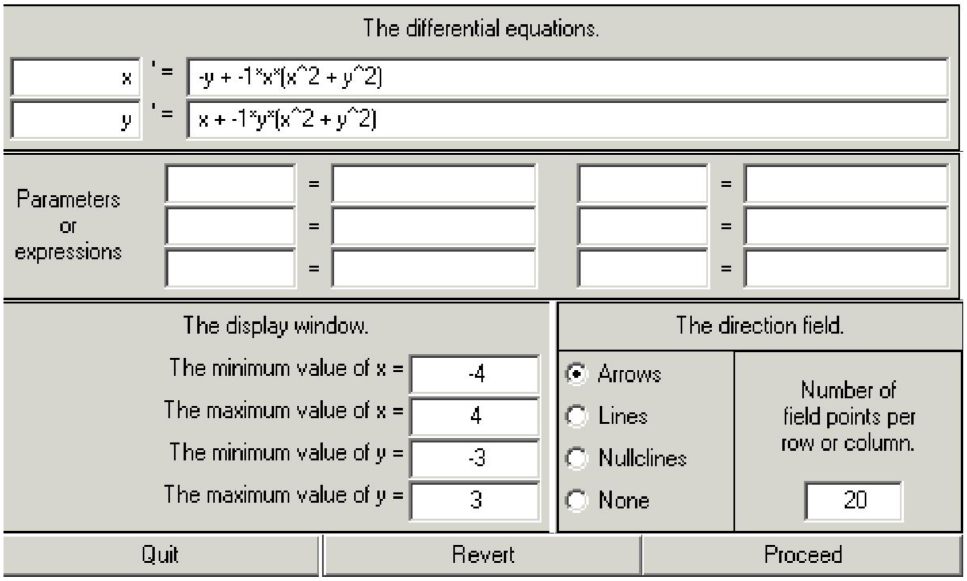
The PPLANE6 program has a very nice additional feature. Take your cursor and click on a point on the PPLANE6 display window. The trajectory from that point will be displayed as a blue line. The dialogue box at the bottom of the display window will indicate the point you clicked on and how the trajectory proceeds from there, both in the forward and backward directions (in time). Take your time to play around with different trajectories. Notice that the initial direction is indicated by the arrows near the point you click on.

A different problem: You may be familiar with the fact that linearization is a process that helps you determine the stability of fixed points–but that it is not always accurate when it predicts a center. We want to use PPLANE6 to help us with such a problem[3]. Consider the system:

$$\begin{aligned} \dot{x} &= -y + ax(x^2 + y^2), \\ \dot{y} &= x + ay(x^2 + y^2) \end{aligned}$$

The linearization predicts that the point $(0,0)$ is always a center. We want to see if this is true or not. We will consider two values of a, $a = -1, 1$. We start with $a = -1$ and we will consider the phase plane over the intervals $-4 \leq x \leq 4$ and $-3 \leq y \leq 3$. In the display window, enter the systems of equations. Remember that * must be used in MATLAB^{TM} to indicate multiplication, and that ˆ needs to be used to indicate exponentiation. When entered correctly, (I am keeping all terms and simply substituting in -1 for a), the window should now look like:

[3]Strogatz, 1994



Once you have entered the equations and changed the intervals for x and y, click "proceed." Click on several points to form the trajectories. Does it appear that $(0,0)$ is a center? If not, what is it? What happens when you try $a = 1$? How do these cases differ from each other? (You can try $a = 0$, but many of the trajectories seem to get stuck due to computer error. However, you can click on the "stop" button to end the computation.)

We have found that the behavior for this system really depends on the value of the parameter a. We will talk more about this type of behavior over the course of PCMI.

Feel free to try more examples. Be creative! (Or use a book!)

References

1. Polking, John C. http://math.rice.edu/ dfield/
2. Strogatz, S. *Nonlinear Dynamics and Chaos.* Perseus Books Publishing, LLC., 1994.