Linearization!

Recall that for a non-linear system,

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y), \end{aligned} \tag{1}$$

we can form the Jacobian matrix **J**:

$$J = \begin{pmatrix} \frac{df}{dx} & \frac{df}{dy} \\ \frac{dg}{dx} & \frac{dg}{dy} \end{pmatrix}.$$

If the system of equations (1) has a fixed point (x^*, y^*), then we can (*usually*) determine the nature of the fixed point via the eigenvalues of **J** evaluated at (x^*, y^*):

$$J(x^*, y^*) = \begin{pmatrix} \frac{df}{dx}(x^*, y^*) & \frac{df}{dy}(x^*, y^*) \\ \frac{dg}{dx}(x^*, y^*) & \frac{dg}{dy}(x^*, y^*) \end{pmatrix}.$$

If all the eigenvalues have non-zero $\mathbb{R}$ part, the fixed point is **hyperbolic**. The linearization is **robust**, i.e. the classification holds, for hyperbolic fixed points. For non-hyperbolic fixed points when at least one eigenvalue has $\mathbb{R}$ part $= 0$, further analysis is needed. Consult the copy from (Strogatz, 1994) for a summary of the nature of fixed points in the linear case.

For You!

Exercise 1.[1] For each of the following systems, find the fixed points, classify them, sketch the neighboring trajectories, and try to fill in the rest of the phase portrait.

(a) $\dot{x} = x - y$; $\dot{y} = x^2 - 4$

(b) $\dot{x} = \sin y$; $\dot{y} = x - x^3$

(c) $\dot{x} = 1 + y - e^{-x}$; $\dot{y} = x^3 - y$

(d) $\dot{x} = y + x - x^3$; $\dot{y} = -y$

(e) $\dot{x} = \sin y$; $\dot{y} = \cos x$

(f) $\dot{x} = xy - 1$; $\dot{y} = x - y^3$

Extra! Generate phase portraits for the systems above to check your classification of the fixed points.

[1] Strogatz, 1994

Exercise 2.[2] Consider the following system of equations describing the interactions between kangaroos, k, and their predators p:

$$\begin{aligned}\dot{k} &= \alpha k - \beta k^2 - \gamma k p \\ \dot{p} &= \sigma p + \lambda k p\end{aligned}$$

Find the fixed points and classify them in the following two cases: (**i.**) $\alpha/\beta < \sigma/\lambda$; (**ii.**) $\alpha/\beta < \sigma/\lambda$.

Exercise 3–Challenge.[3] The system

$$\begin{aligned}\dot{x} &= -y - x^2 \\ \dot{y} &= x\end{aligned}$$

has a linear center at $(0,0)$ (*Verify this!*) Show that it is a non-linear center. (Note: This requires more analysis. Consult (Strogatz, 1994) if you don't know how to do it, but want to!)

References

1. Strogatz, S. *Nonlinear Dynamics and Chaos.* Perseus Books Publishing, LLC., 1994.

[2] Taubes, 2001
[3] Strogatz, 1994