de Pillis | Gallegos
Tumor Modeling Course
Solutions to Tues 7/5 Worksheet

Nondimensionalization Exercises - Solutions!

1. $\frac{dN}{dt} = rN(1 - N/K)$

N is the population we are tracking - it could be pure numbers or a population density (#/area for example)

(a) We can figure out the units because we know the units on the left hand side must match the units on the right hand side

$$\frac{\text{population}}{\text{time}} = r \cdot \text{population} - \frac{r \cdot \text{population}^2}{K}$$

r must have units 1/time (which makes sense since it's a rate!)

then the last term on the right hand side is

$\frac{\text{population}^2}{\text{time} \cdot K}$ so K must have units of population (which makes sense since it is a carrying capacity!) and No has units of population (#, or #/area) just like N.

(b) Now were choosing scales. x is our dimensionless population size. We can choose $x = N/K$. Since both N and K have dimensions of population, x is dimensionless.

We can choose $\tau = tr$. Since t has units of time and r has units of 1/time, τ is dimensionless.

We can choose $x_0 = N_0/K$. As above, x_0 is dimensionless.

Now we substitute in: $N = Kx$, $t = \tau/r$, $N_0 = Kx_0$:

$$\frac{dN}{dt} = \frac{d(Kx)}{d(\frac{1}{r}\tau)} = \frac{K\,dx}{1/r\;d\tau} = \cancel{rK}\frac{dx}{d\tau} = \cancel{rK}x\left(1 - \frac{\cancel{K}x}{\cancel{K}}\right)$$

$$\Rightarrow \frac{dx}{d\tau} = x(1-x)$$

and $N(0) = \cancel{K}X(0) = N_0 = KX_0 \Rightarrow X(0) = X_0$

So our new dimensionless system is:

$$\frac{dX}{d\tau} = X(1-X) \qquad X(0) = X_0$$

(c) In terms of our original equation, we now rescale so that $u = N/N_0$ and $u_0 = N_0/N_0 = 1$.

Then we have (since $N = uN_0$)

$$\frac{dN}{dt} = \frac{d(uN_0)}{d(t/r)} = \frac{N_0}{1/r}\frac{du}{d\tau} = rN_0u\left(1 - \frac{N_0u}{K}\right)$$

$$\Rightarrow \frac{du}{d\tau} = u\left(1 - \frac{u}{K}\right), \quad u_0 = 1$$

* notice in this case we still have the parameter K.

(d) The n.d. in (b) gets rid of all parameters and measures population w.r.t. the carrying capacity. If the population at time 0 differs greatly from the carrying capacity you could have very large or small values which may cause computational problems. The n.d. in (c) scales the population to order 1 (although u/K may still be large or small as above). The advantage is that population is measured wrt initial population size instead of carrying capacity

2. $$\frac{dT}{dt} = qe^{-\theta/T} - k(T - T_f)$$

(a) Again we can answer this question by comparing units on each side (see next page)

the 1 indicates $e^{-\theta/T}$ has to be unitless.

$$\frac{\text{temperature}}{\text{time}} = q \cdot 1 - k\,(\text{temp.} - \text{temp})$$

So q has to have units of temp/time.
Since $e^{-\theta/T}$ is unitless θ has to have the same units as T ; θ has units of temperature.
Then, k has to have units of 1/time.
(Note that temperature may be measured in degrees F, or degrees C)

(b) We have the temperature scale given. If $\tilde{T}$ is our n.d. temperature, then $\tilde{T} = T/T_f$. Not choosing a specific time scale, let n.d time $\tau = t/\hat{t}$. We'll plug these in and see what we might need $\hat{t}$ to be.

$$\frac{dT}{dt} = \frac{d(T_f \tilde{T})}{d(\hat{t}\tau)} = q \cdot e^{-\theta/T_f\tilde{T}} - k\,(T_f\tilde{T} - T_f)$$

$$\frac{T_f}{\hat{t}}\frac{d\tilde{T}}{d\tau} = q\,e^{(-\frac{\theta}{T_f})\tilde{T}} - kT_f\,(\tilde{T} - 1)$$

$$\frac{d\tilde{T}}{d\tau} = \frac{q\hat{t}}{T_f}\,e^{(-\theta/T_f)\tilde{T}} - k\hat{t}\,(\tilde{T} - 1)$$

Now we choose $\hat{t} = T_f/q$ which has units of time (✓) since temp/temp/time = time, based on the following assumptions: (1) T_f must be very large (since it's the furnace temperature) ; (2) $T_f \gg q$ (3) k is not small enough to "outdo" the "largeness" of T_f. Then:

$$\frac{d\tilde{T}}{d\tau} = e^{-b\tilde{T}} - a(\tilde{T} - 1) \;;\; \tilde{T}(0) = \tilde{T}_0$$

where $b = \theta/T_f$ and $a = kT_f/q$. The biggest $e^{-b\tilde{T}}$ can be is 1 and a is large! ($\tilde{T}_0 = T_0/T_f$)