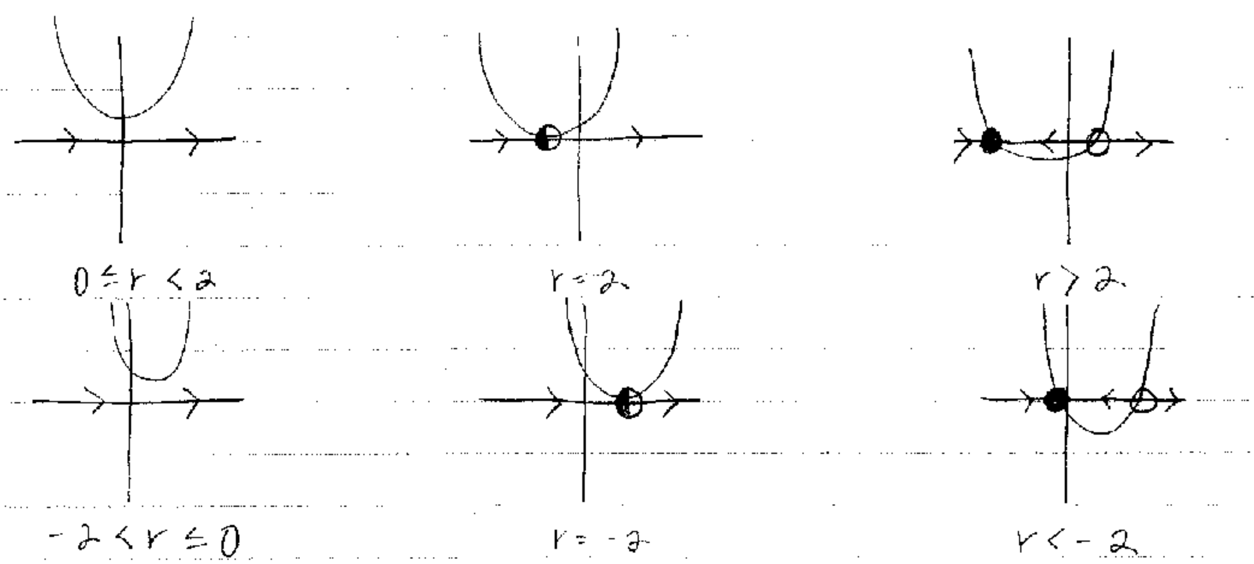
dePillis and Gallegos
Tumor Modeling course
Bifurcation Solutions

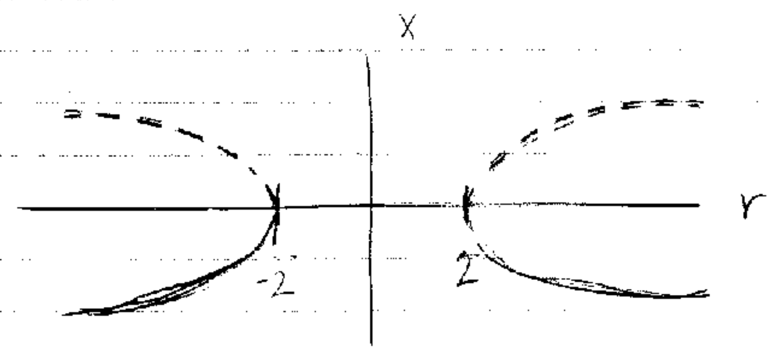
DAY 7

BIFURCATIONS! SOLUTIONS!

1. $\dot{x} = 1 + rx + x^2$

From the quadratic formula, the roots of $x^2 + rx + 1 = 0$ are $\frac{-r \pm \sqrt{r^2 - 4}}{2}$. There are no real roots for $r \in (-2, 2)$; there is one real root when $r = \pm 2$ and there are 2 real roots for $|r| > 2$.





there is a saddle node bifurcation on either side of the X axis.

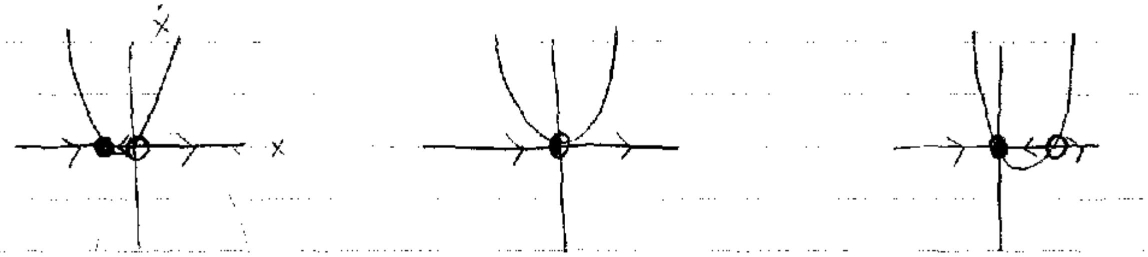
2. $\dot{x} = x - rx(1-x)$

first I solve for r_c (the critical value of r) by requiring: $x = rx(1-x)$ (1)

and $(x)' = [rx(1-x)]'$ or $1 = rx(-1) + (1-x)r$ (2)

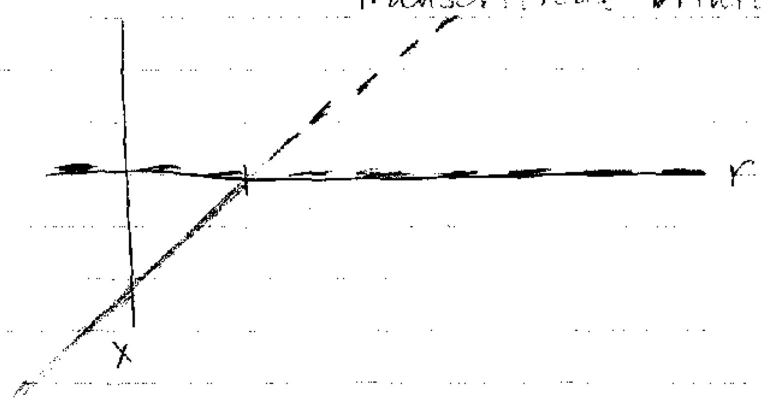
Solving (1) for r gives $r = 1/(1-x)$. Plugging that in to (2) gives $x = 0 \Rightarrow \boxed{r_c = 1}$

Then, here are some characteristic fields.



$r < 1$ (used $r = .5$) $\quad$ $r = 1$ $\quad$ $r > 1$ (used $r = 3$)

transcritical bifurcation.



3. $\dot{x} = x + \dfrac{rx}{1+x^2}$

Again, we can solve for r_c first using:

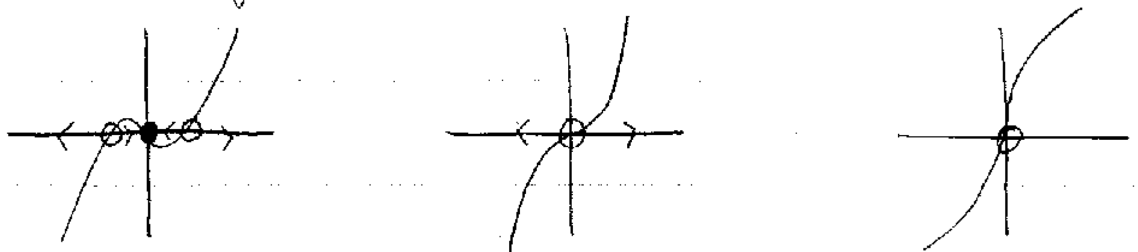
(1) $-x = rx/(1+x^2)$

(2) $[-x]' = [rx/(1+x^2)]'$

from (1) we get $r = -(1+x^2)$ or $x^2 = -r-1$

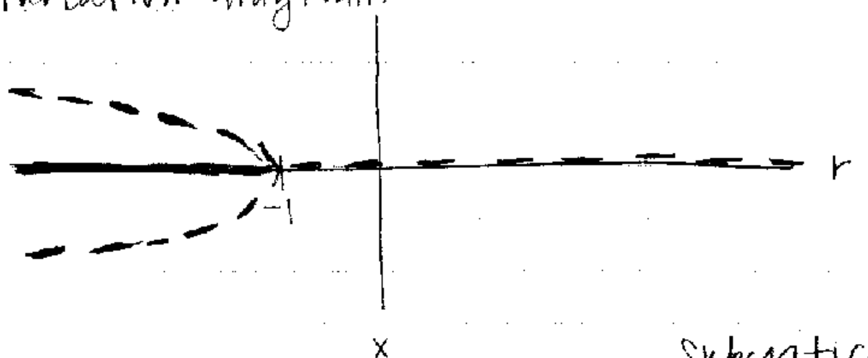
(2), explicitly, is: $-1 = \dfrac{r(1+x^2) - (rx)(2x)}{(1+x^2)^2}$

Solving this gives $r_c = -1$.



$r < -1$ (used $r=-2$) | $r = -1$ | $r > -1$ (used $r=1$)

bifurcation diagram:



subcritical pitchfork

4. $\dot{N} = rN(1 - N/K) - HN/(A+N)$

(a) When $N = A$ the fish are harvested at ½ the maximal rate. So A is sort of a relative measure of the size of the fish population.

(b) To nondimensionalize we need to rescale N and t. We can think of $N = \tilde{N}\bar{N}$ ($\tilde{N}$ is dimensionless, $\bar{N}$ is the dimensional scale) and $t = \tau\bar{t}$ (τ is dimensionless, $\bar{t}$ is the dimensional scale). Our parameters are r, K, H and A. We know K has units of population – so we choose it for $\bar{N}$. We will choose $\bar{t}$ later to obtain the desired formulation.

LHS $\quad \dfrac{dN}{dt} = \dfrac{d(\tilde{N}\bar{N})}{d(\tau\bar{t})} = \dfrac{d(\tilde{N}K)}{d(\tau\bar{t})} = \dfrac{K}{\bar{t}}\dfrac{d\tilde{N}}{d\tau}$

RHS $\quad rN(1 - N/K) - HN/(A+N)$

$$= r\tilde{N}K(1 - \tilde{N}\cancel{K}/\cancel{K}) - H\tilde{N}K/(A + \tilde{N}K)$$

$$= K\left[r\tilde{N}(1-\tilde{N}) - H\tilde{N}/(A + \tilde{N}K)\right]$$

so $\quad \dfrac{\cancel{K}}{\bar{t}}\dfrac{d\tilde{N}}{d\tau} = \cancel{K}\left\{r\tilde{N}(1-\tilde{N}) - \dfrac{H\tilde{N}}{A+\tilde{N}K}\right\}$

$$\frac{d\tilde{N}}{d\tau} = \bar{t}r\tilde{N}(1-\tilde{N}) - \frac{\bar{t}H\tilde{N}}{A+\tilde{N}K} \qquad (*)$$

we want $\bar{t}r = 1$ so choose $\bar{t} = 1/r$ (check units! Is this okay?!!) we also want to rescale the denominator of the harvesting term. So multiply the whole fraction by $1/K \,/\, 1/K$:

$$\frac{(\bar{t}H/K)\tilde{N}}{A/K + \tilde{N}}$$

Then (*) becomes

$$\frac{d\tilde{N}}{d\tau} = \tilde{N}(1-\tilde{N}) - \frac{h\tilde{N}}{a+\tilde{N}}$$

for $h = H/Kr$ and $a = A/K$.

4 cont'd.

(c) We know when $dN/dt = 0$, $N(1-N) = hN/(a+N)$

The fixed points are the roots of the equation:

$$N(N^2 + (a-1)N + (h-a)) = 0$$

$N=0$ is always a fixed point. The other fixed points satisfy

$$\frac{-(a-1) \pm \sqrt{(a-1)^2 - 4(h-a)}}{2}$$

When the value under the radical is positive, the system has 3 real fixed points. When the value under the radical is 0 the system has 2 real fixed points. For a & h s.t. the radical is negative, there is only the fixed point 0.

The critical value of a (at which the bifurcation occurs) is the value a_c s.t. $(a-1)^2 - 4(h-a) = 0$. Since a is positive, this occurs when $a = -1 + 2\sqrt{h}$.

$\tilde{N} = 0$ is an unstable fixed point

When there are 2 fixed points, 0 is unstable and $(1-a)/2$ is stable

When there are 3 fixed points 0 is unstable and the other 2 are stable.

(d) $\frac{d\tilde{N}}{dt} = \tilde{N}(1-\tilde{N}) - h\tilde{N}/a+\tilde{N} = f(\tilde{N})$

Consider $f'(\tilde{N})$: $\tilde{N}(-1) + (1-\tilde{N})(1) - \left\{ \frac{(a+\tilde{N})h - h\tilde{N}(1)}{(a+\tilde{N})^2} \right\}$

$f'(0) = 1 - \frac{ah}{a^2}$

So when $a = h$ $f'(0) = 0$ so that 0 is semistable.

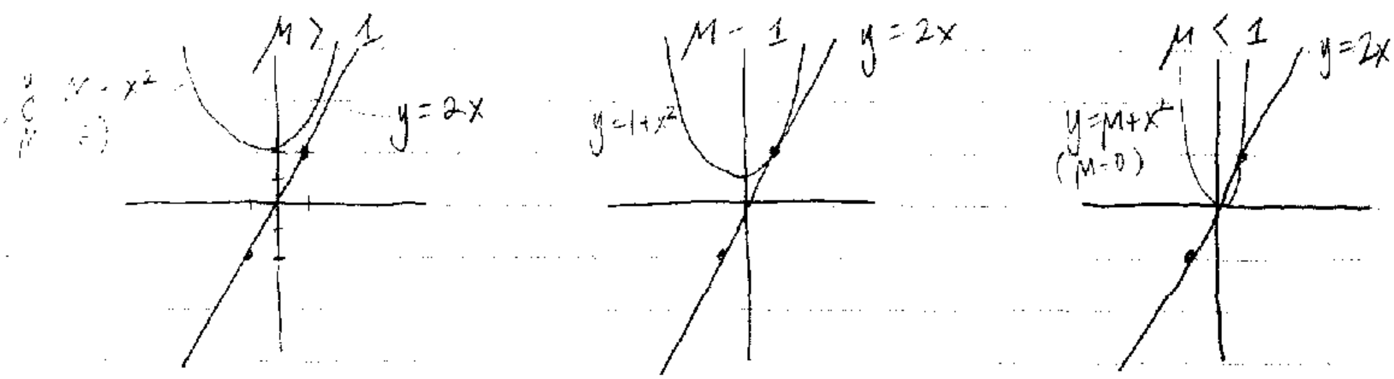
If $a > h$ or $a < h$ the stability changes.

This is transcritical.

(e) I already showed that a_c occurs above. This is a saddle node bifurcation.

5. $\dot{x} = y - 2x$; $\dot{y} = \mu + x^2 - y$

(a) The x nullcline is given by $y = 2x$. The y nullcline is given by $y = \mu + x^2$. The appearance of the nullclines (and fixed points) depends on μ. There are fixed points when $2x = \mu + x^2$ or $x^2 - 2x + \mu = 0$. This has one real root for $\mu = 1$ – this is μ_c (the bifurcation point). Otherwise $x = \frac{2 \pm \sqrt{4 - 4\mu}}{2}$ which has no real roots when $\mu > 1$ (ie no fixed points) and 2 real roots for $\mu > 1$. So there are 3 nullcline cases



(b) The bifurcation occurs at $\mu_c = 1$. It is a saddle node bifurcation.

(c) see attached phase portraits.

6. $\dot{x} = y$
$\dot{y} = \mu y + x - x^2 + xy$

(a) $J = \begin{bmatrix} 0 & 1 \\ 1-2x+y & \mu + x \end{bmatrix}$

$J(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & \mu \end{bmatrix}$ w/ evalues $\dfrac{\mu \pm \sqrt{\mu^2+4}}{2}$

But $\sqrt{\mu^2+4} > \mu$ no matter what μ is. So $\frac{1}{2}(\mu + \sqrt{\mu^2+4}) > 0$ and $\frac{1}{2}(\mu - \sqrt{\mu^2+4}) < 0$, i.e. the evalues are real with opposite signs and so $(0,0)$ is a saddle point for all μ.

(b) Any fixed point must have $y=0$ (x nullcline) so we look at the y nullcline when $y=0$: $x - x^2 = 0$
$x(1-x) = 0$

So the other fixed point is $(1,0)$.

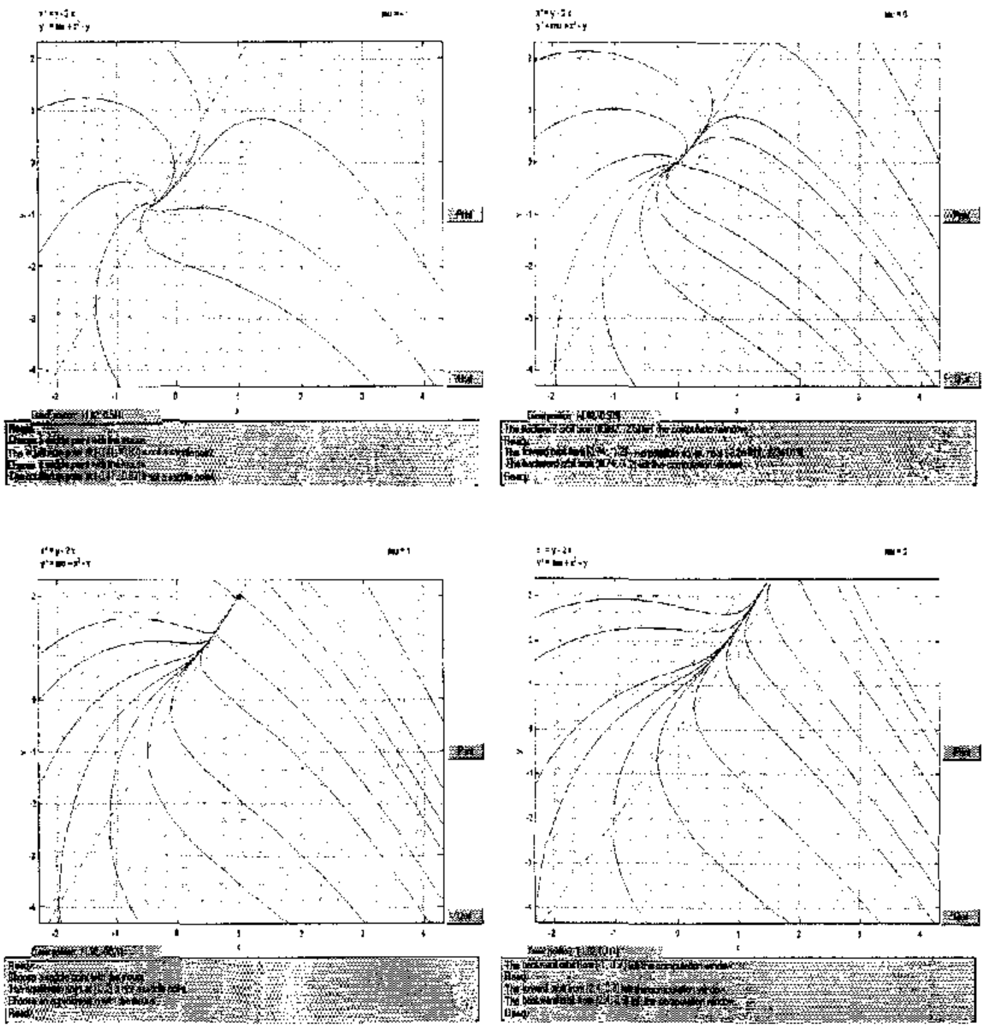
$J(1,0) = \begin{bmatrix} 0 & 1 \\ -1 & \mu+1 \end{bmatrix}$ w/ evalues $\dfrac{(\mu+1) \pm \sqrt{(\mu+1)^2-4}}{2}$

When $\mu = 1$, there is the repeated evalue 1 $(= \frac{2}{2})$ and $(1,0)$ is an unstable node. When $\mu > 1$ or $\mu < -2$, there are 2 real evalues which can have opposite sign. When $-2 < \mu < -1$ the Re part of the evalues are negative and so the fixed point is a stable spiral. For $-1 < \mu < 1$ the Re parts are positive so $(1,0)$ is an unstable spiral. At $\mu = -1$ the linearization predicts a center. This would need to be checked more.

(c) & (e) See attached plots
(d) From Maple, $\mu_c \approx -.8645$

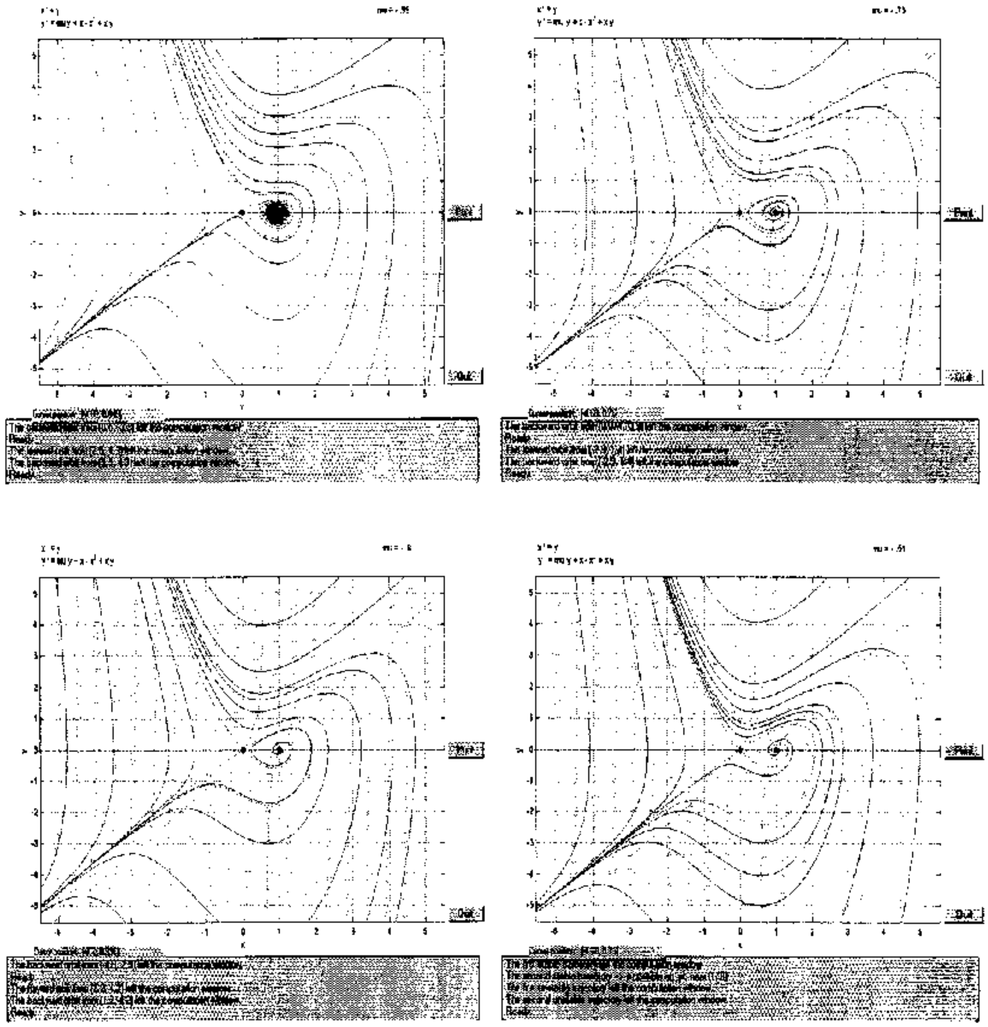
Bifurcation Exercises Solutions: Computer Generated Plots.

Exercise 5c. The following plots are generated from pplane6 using $\mu = -1, 0, 1, 2$.



(9)

Exercise 6c. The following plots were generated from pplane6 using $\mu = -.99, -.75, -.6, -.51$.



Exercise 6e. The following plots were generated from pplane6 using $\mu = -.9, -.8645, -.8$.