

Bifurcations!—Exercises for You.

*Exercise 1.*¹ For the following system, sketch all the qualitatively different vector fields that occur as r is varied. Show that a saddle-node bifurcation occurs at a critical value of r , to be determined. Finally sketch the bifurcation diagram of fixed points x^* versus r .

$$\dot{x} = 1 + rx + x^2$$

*Exercise 2.*¹ For the following system, sketch all the qualitatively different vector fields that occur as r is varied. Show that a transcritical bifurcation occurs at a critical value of r , to be determined. Finally sketch the bifurcation diagram of fixed points x^* versus r .

$$\dot{x} = x - rx(1 - x)$$

*Exercise 3.*¹ For the following system, sketch all the qualitatively different vector fields that occur as r is varied. Show that a pitchfork bifurcation occurs at a critical value of r , to be determined. Finally sketch the bifurcation diagram of fixed points x^* versus r .

$$\dot{x} = x + \frac{rx}{1 + x^2}$$

*Exercise 4.*¹ **A fishery model.** A model of a fishery assumes that in absence of harvesting, fish grow logistically. The harvesting rate is designated by H and the model for the fish population, N is:

$$\frac{dN}{dt} = rN \left(1 - \frac{N}{K} \right) - H \frac{N}{A + N}$$

where $H > 0$ and $A > 0$.

(a.) Give a biological interpretation of the parameter A ; what does it measure?

(b.) Show that the system can be rewritten in dimensionless form as

$$\frac{dx}{d\tau} = x(1 - x) - h \frac{x}{a + x},$$

for suitably defined dimensionless quantities, x, τ, a and h .

¹Strogatz, 1994

(c.) Show that the system can have one, two, or three fixed points, depending on the values of a and h . Classify the stability of the fixed points in each case.

(d.) Analyze the dynamics near $x = 0$ and show that a bifurcation occurs when $h = a$. What type of bifurcation is it?

(e.) Show that another bifurcation occurs when $h = \frac{1}{4}(a + 1)^2$, for $a < a_c$, where a_c is to be determined. Classify this bifurcation.

*Exercise 5.*¹ Consider the system

$$\dot{x} = y - 2x; \quad \dot{y} = \mu + x^2 - y.$$

(a.) Sketch the nullclines.

(b.) Find and classify the bifurcations that occur as μ varies.

(c.) Sketch the phase portrait as a function of μ .

*Exercise 6.*¹ Consider the system,

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= \mu y + x - x^2 + xy.\end{aligned}$$

(a.) Show that the origin is a saddle point for all values of μ .

(b.) Find any other equilibria and determine their stability.

(c.) Numerically plot phase portraits for values of μ between -1 and -0.5 . What happens to the unstable manifold through the origin as μ is varied?

(d.) Numerically find the critical value of μ at which the stable and unstable manifolds through the origin intersect. (*This is called a homoclinic connection or homoclinic orbit.*)

(e.) Plot a few phase portraits for μ -values above and below this critical value and describe what happens to the unstable manifold through the origin.

This type of bifurcation is called a *homoclinic bifurcation*.

References

1. Strogatz, S. *Nonlinear Dynamics and Chaos*. Perseus Books Publishing, LLC., 1994.