MATH 25B FALL 2010: CALCULUS LECTURE 8

DAGAN KARP

ABSTRACT. In this note, we explore and prove both parts of the fundamental theorem of calculus.

1. FTOC

Now that we have defined the definite integral of a function from on a closed interval, a natural question to ask is: For which functions does this integral exist? That is, which functions are integrable?

Theorem 1. *Every continuous function is integrable.*

Remark 2. Many discontinuous functions are also integrable!

Example 3. *Let* f *be given by*

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Here, for every partition P *of* $[a, b]$, $L(f, P) = 0$ *and* $U(f, P) = b - a$.

2. FUNDAMENTAL THEOREM OF CALCULUS

Recall from your previous studies of calculus, the area A under the graph of the (non-negative) function $y = f(x)$ can be computed via anti-derivation. Specifically, find any function $F(x)$ such that $F'(x) = f(x)$. Then the area A is $F(b) - F(a)$. Why the *%&$ does this work?

Example 4. *To compute* $\int_1^3 x^2 dx$, *we note that* $F(x) = x^3/3$ *is an antiderivative, and thus* $A = F(3) - F(1) = 27/3 - 1/3 - 26/3$. *This is denoted*

$$A = \int_1^3 x^2 dx = \frac{x^3}{x}|_1^3 = \frac{26}{3}$$

Let's try to understand why this works? Let $f(t)$ be any integrable function and fix the number $a \in \mathbb{R}$. Define the *area function* by

$$A(x) = \int_a^x f(t) dt$$

which gives us the area under the curve from a to x. Note that $A(a) = 0$.

Proposition 5. *If* $f(t)$ *is continuous, then* $A(x)$ *is differentiable.*

Date: November 4, 2010.

Proof.

$$\frac{d}{dx}A(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h}.$$

Note that $A(x+h) - A(x)$ is the area under the graph of f from x to $x+h$. Define

$$M_h = \text{Max}\{f(t) : t \in [x, x+h]\}$$
$$m_h = \min\{f(t) : t \in [x, x+h]\}$$

Then

$$m_h \cdot h \leqslant A(x+h) - A(x) \leqslant M_h \cdot h,$$

and therefore

$$m_h \leqslant \frac{A(x+h) - A(x)}{h} \leqslant M_h$$

and hence

$$\lim_{h\to 0} m_h \leqslant \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} \leqslant \lim_{h\to 0} M_h$$

Now, since f is continuous, we have

$$\lim_{h\to 0} m_h = f(x) = \lim_{h\to 0} M_h$$

Thus, we conclude

$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = f(x)$$

We have just proved FTOC I.

Theorem 6 (Fundamental Theorem of Calculus I)**.** *If* f *is continuous on* $[a, b]$*, then*

$$A(x) = \int_a^x f(t)dt$$

is differentiable and

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x).$$

The second FTOC returns to our original question.

Theorem 7 (Fundamental Theorem of Calculus II)**.** *If* f *is continuous of* $[a, b]$ *and* $f(x) = F'(x)$ *for some function* F*, then*

$$\int_a^b f(t)dt = F(b) - F(a).$$

Proof. Let $A(x) = \int_a^x f(t)dt$. Then our goal is to show that $A(b) = F(b) - F(a)$. From FTOC I, we have

$$A'(x) = f(x) = F'(x),$$

where the last equality is by hypothesis. So, by corollary of MVT,

$$A(x) = F(x) + C$$

for some constant C. To determine C, note that

$$0 = A(a) = F(a) + C \Rightarrow C = -F(a).$$

Therefore

$$A(x) = F(x) - F(a)$$

and thus

$$A(b) = F(b) - F(a).$$

□

Example 8.

$$\frac{d}{dx}\int_3^x \cos^7 t dt = \cos^7 x$$

Example 9.

$$\int_a^b x^n dx = \frac{x^{n+1}}{n+1}|_a^b = \frac{b^{n+1} - a^{n+1}}{n+1}.$$

Note that this works for $n \neq -1$. So what can we say about this exception? That is, can we find a function whose derivative is $1/x$?

Indeed, we can use our newly rediscovered knowledge of the FTOC I to *construct* a function whose derivative is $1/x$. Consider the area function

(1) $$A(x) = \int_a^x \frac{1}{t}dt$$

which measures the area under the graph of the function $f(t) = 1/t$ from a to x. If $f(t)$ satisfies the conditions of the FTOC, then we may conclude that

$$\frac{d}{dx}(A(x)) = \frac{d}{dx}\left(\int_a^x \frac{1}{t}dt\right) = f(x) = \frac{1}{x}.$$

Well, the function $f(t) = 1/t$ does not satisfy the hypothesis of FTOC I, namely, $1/t$ is not everywhere continuous. As M.J. has reminded us, we can't always get what we want, but here again we find that we get what we need. The function $1/t$ is not *everywhere* continuous, but if we restrict our attention to strictly positive values of t, i.e. $t > 0$, then $f(t)$ is indeed continuous on that restricted domain, and hence FTOC I applies.

Thus, define $A(x)$ as above (1), with the requirement that both a and x are positive. Then by FTOC I, $A'(x) = 1/x$, and our search is complete.

However, I remain slightly unsatisfied: our solution is not unique. For any $a > 0$, we have the area function (1). So, let us once and for all choose a special value for a: choose

$a = 1$. Aside from the fact that the number 1 is very nice in many ways, we'll justify this choice as we consider the properties of this function we've just nailed down.

Definition 10. *The* natural logarithm *function, denoted* $\ln(x)$, *is defined by*

$$\ln(x) = \int_1^x \frac{1}{t} dt.$$