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Math 131 Notes

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Sequences

Recall 1. A sequence $\{p_n\}$ in X is a function $f : \mathbb{N} \to X$ such that $f(n) = p_n \in X$.

Definition 1. The sequence $\{p_n\}$ *converges* in X if there exists $p \in X$ such that for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > N$ then $d(p, p_n) < \epsilon$. We say $\{p_n\}$ *converges to* p. Or equivalently, p is the *limit of* $\{p_n\}$.

Notation 1. $p_n \to p \iff \lim_{n\to\infty} \{p_n\} = p$.

Definition 2. If $\{p_n\}$ does not converge to any point in X then it *diverges*.

Definition 3. The *range* of $\{p_n\}$ is:

$$\{x \in X | x = p_n \text{ for some } n\}.$$

Definition 4. We say $\{p_n\}$ is *bounded* in X if the range of $\{p_n\}$ is bounded in X.

True/False Questions:

(a) $p_n \to p, p_n \to p' \Rightarrow p = p'$. True.

(b) $\{p_n\}$ is bounded $\Rightarrow p_n$ converges. False.

(c) p_n converges $\Rightarrow \{p_n\}$ is bounded. True.

(d) $p_n \to p \Rightarrow p$ is a limit point of the range of $\{p_n\}$. False.

(e) p is a limit point of $E \subseteq X$, then there exists some sequence $\{p_n\}$ such that $p_n \to p$. True.

(f) $p_n \to p \iff$ Every neighborhood of p contains all but finitely many terms in $\{p_n\}$. True.

Remark 1. "All but finitely many" is equivalent to "Almost all."

Proof. a) Let $\epsilon > 0$. Then there exists $N_1 \in \mathbb{N}$ such that $n > N_1$, $d(p, p_n) < \frac{\epsilon}{2}$. Similarly there exists $N_2 \in \mathbb{N}$ such that $n > N_2$, $d(p', p_n) < \frac{\epsilon}{2}$. Then for $n > \max(N_1, N_2)$:

$$d(p, p') \leq d(p, p_n) + d(p', p_n) < 2\frac{\epsilon}{2} = \epsilon.$$

Thus $d(p, p') < \epsilon$ for all $\epsilon > 0$. Thus $d(p, p') = 0$. Hence $p = p'$.

b) Consider Ex. (4) where there is oscillation between two points.

c) Suppose $p_n \to p$. Since $1 > 0$ (our choice of ϵ), there exists $N \in \mathbb{N}$ such that for $n > N$ implies $d(p_n, p) < 1$. Let $r = \max(1, d(p, p_1), \ldots, d(p, p_n))$. Then $p_i \in B(p, r)$ for all $i \in \mathbb{N}$.

d) Consider Ex. (3) where the sequence is simply one point.

e) $p \in E'$. For all $n \in \mathbb{N}$ such that $p_n \in E$ such that $d(p_n, p) < 1/n$ and $p_n \neq p$. There for a sequence $\{p_n\}$. Let $\epsilon > 0$. Choose $N > \frac{1}{\epsilon}$. Then for each $n > N$, we have $d(p, p_n) < \frac{1}{n} < \frac{1}{N} < \epsilon$. Thus this sequences $\{p_n\}$ converges to p as desired.

f) ($\Rightarrow$) Suppose $p_n \to p$. Given $B(p, \epsilon)$, there exists $N \in \mathbb{N}$ such that when $n > N$, it follows that $d(p, p_n) < \epsilon$, i.e. $p_n \in B(p, \epsilon)$, leaving only finitely many points, p_1 through p_n possible.

($\Leftarrow$) For all $\epsilon > 0$, $B(p, \epsilon)$ contains almost all $\{p_n\}$. For $\epsilon > 0$, let

$$m = \max\left(n \in \mathbb{N} | p_n \notin B(p, \epsilon)\right).$$

Then $n > m$ implies $p_n \in B(p, \epsilon)$, i.e. $d(p_n, p) < \epsilon$. □

Theorem 1. *(Limit Laws) Let $\{s_n\}, \{T_n\}$ be sequences in $\mathbb{C}$ and $s_n \to s$ and $t_n \to t$. Then the following hold:*

- $\lim_{n \to \infty} (s_n + t_n) = s + t$.
- $\lim_{n \to \infty} (cs_n) = cs$ *for all* $c \in \mathbb{C}$.
- $\lim_{n \to \infty} (c + s_n) = c + s$ *for all* $c \in \mathbb{C}$.
- $\lim_{n \to \infty} (s_n t_n) = st$ *for all* $c \in \mathbb{C}$.
- $\lim_{n \to \infty} \dfrac{1}{s_n} = \dfrac{1}{s}$ *for all* $c \in \mathbb{C}$.

Proof. a) Idea: $\boxed{|(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t|}$. Let $\epsilon > 0$. Then there exists $N_1 \in \mathbb{N}$ such that $n > N_1$ implies $d(s_n, s) < \epsilon/2$ and there exists $N_2 \in \mathbb{N}$ such that $n > N_1$ implies $d(t_n, s) < \epsilon/2$. For $n > \max(N_1, N_2)$,

$$\begin{aligned} d(s_n + t_n, s + t) &= |(s_n + t_n) - (s + t)| \\ &= |s_n - s| + |t_n - t| \\ &< \epsilon. \end{aligned}$$

b) Idea: $|cs_n - cs| \leq c|s_n - s|$.

c) Idea: $|s_n t_n - st| = |(s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)|$. Let $\epsilon > 0$ Choose $k = \max(s, t, 1, \epsilon)$. Then there exists N_1, N_2 such that $n > N_1$ implies $d(s_n, s) < \dfrac{\epsilon}{3k}$ and similarly if $n > N_2$ implies $d(t_n, t) < \dfrac{\epsilon}{3k}$. Let $N = \max N_1, N_2$ For $n > N$ we know:

$$\begin{aligned} |s_n t_n - st| &= |(s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)| \\ &= \frac{\epsilon^2}{9k^2} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \frac{\epsilon}{9k} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &< \epsilon. \end{aligned}$$

□