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MATH 131 NOTES

Definition 1. A partition P of $[a,b] \subseteq \mathbb{R}$ is a set $P = \{x_0, \ldots, x_n\}$ such that $a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$. Notice that this is monotonically increasing but not necessarily strictly increasing. We denote $\Delta x_i = x_i - x_{i-1}$. The *mesh* (or *norm*) is $\text{mesh}(p) = \max\{\Delta x_i | i = 1, \ldots, n\}$.

Definition 2. Another partition $Q = \{y_0, \ldots, y_m\}$ on $[a,b]$ is a refinement of P is $P \subseteq Q$.

Definition 3. Let f be bounded on $[a,b]$. Let P be a partition. Define

$$m_i = \inf\{f(x) | x \in [x_{i-1}, x_i]\},$$
$$M_i = \sup\{f(x) | x \in [x_{i-1}, x_i]\}.$$

The *upper Darboux sum* and *lower Darboux sum* of f on $[a,b]$ with respect to P are

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i,$$
$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i.$$

Definition 4. The *upper and lower Darboux integrals* of f over $[a,b]$ are

$$\overline{\int_a^b} f(x)dx = \inf\{U(P,f) | \text{partitions } P \text{ on [a,b]}\},$$
$$\underline{\int_a^b} f(x)dx = \sup\{U(P,f) | \text{partitions } P \text{ on [a,b]}\}.$$

Definition 5. If $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$ we say f is Darboux integrable and we write $\int_a^b f(x)dx$.

Remark 1 (Reimann Sum). Take a single point in each partition and use the function value at that point to compute the approximate area.

Definition 6. A *Riemann sum* of f over $[a,b]$ with respect to P is

$$R(P,f) = \sum_{i=1}^{n} f(c_i) \Delta x_i,$$

with $c_i \in [x_{i-1}, x_i]$.

Definition 7. For $s \in \mathbb{R}$, we say the *Reimann integral* of f over $[a,b]$ is equal to s if for all $\epsilon > 0$ there exists $\delta > 0$ such that $\text{mesh}(p) < \delta \iff |R(P,f) - s| < \epsilon$. In this case, we write $\int_a^b f(x)dx = s$.

Theorem 1. *Let f be bounded on $[a,b]$. Then f is Darboux integrable if and only if f is Riemann integrable.*

Note 1. Let $\mathfrak{R}$ denote the set of all Riemann integrable function.

Definition 8. Let f be bounded on $[a,b]$. Let α be monotonically increasing on $[a,b]$. Given a partition P, we write $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \geq 0$. Then the *upper and lower Riemann-Stieltjes* sums of f over $[a,b]$ with respect to P and α are:

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i,$$
$$L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i.$$

Definition 9. The upper and lower Riemann-Stieltjes integrals of P over $[a,b]$ are:

$$\overline{\int_a^b} f d\alpha = \inf\{U(P,f,\alpha) | \text{partitions } P \text{ on [a,b]}\},$$
$$\underline{\int_a^b} f d\alpha = \inf\{L(P,f,\alpha) | \text{partitions } P \text{ on [a,b]}\}.$$

If $\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha$, then we say f is Riemann-Stieltjes integrable with respect to α over $[a,b]$ and we write $\int_a^b f d\alpha$.

Note 2. The set of all Riemann-Stieltjes integrable functions with respect to α is denoted $\mathfrak{R}(\alpha)$.

Example 1. If $\alpha(x) = 0$, then $\int_a^b f(x)dx = 0$.

Example 2. If $\alpha(x) = x$, then $\int_a^b f d\alpha = \int_a^b f(x)dx$.

Remark 2. We can regard the auxiliary function α as weighting of each partition.

Theorem 2. *If Q is a refinement of P, then"*

$$L(P,f,\alpha) \leq L(Q,f,\alpha),$$
$$U(Q,f,\alpha) \leq U(P,f,\alpha).$$

Proof. Suppose $Q = P \cup x^*$ with $x^*(x_{i-1}, x_i)$. Let:

$$w_1 = \inf\{f(x) | x \in [x_{i-1}, x^*]\},$$
$$w_2 = \inf\{f(x) | x \in [x^*, x_i]\}.$$

Notice that $w_1, w_2 \geq m_i$. Thus:

$$\begin{aligned} L(Q,f,\alpha) - L(P,f,\alpha) &= w_1[\alpha(x^*) - \alpha(x_{i-1}) + w_2[\alpha(x_i) - \alpha(x^*) - m_i\Delta\alpha_i \\ &= (w_1 - m_i)[\alpha(x^*) - \alpha(x_{i-1})] + (w_2 - m_i)[\alpha(x_i) - \alpha(x^*)] \\ &\geq 0. \end{aligned}$$

The proof is completed by iterating this if Q has more than one more point than P. ∎